Immersed-Interface Finite-Element Methods for Elliptic Interface Problems with Non-homogeneous Jump Conditions

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Abstract

In this work, a class of new finite-element methods, called immersed-interface finite-element methods, are developed to solve elliptic interface problems with non-homogeneous jump conditions. Simple non-body-fitted meshes are used. A single function that satisfies the same non-homogeneous jump conditions is constructed using a level-set representation of the interface. With such a function, the discontinuities across the interface in the solution and flux are removed; and an equivalent elliptic interface problem with homogeneous jump conditions is formulated. Special finite-element basis functions are constructed for nodal points near the interface to satisfy the homogeneous jump conditions. Error analysis and numerical tests are presented to demonstrate that such methods have an optimal convergence rate. These methods are designed as an efficient component of the finite-element level-set methodology for fast simulation of interface dynamics that does not require re-meshing.

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1 Introduction

We consider numerical solution of the elliptic interface problem

\[
\begin{align*}
- \nabla \cdot \beta^- \nabla u &= f \quad \text{in } \Omega^-, \\
- \nabla \cdot \beta^+ \nabla u &= f \quad \text{in } \Omega^+, \\
[u]_\Gamma &= w, \\
\left[ \beta \frac{\partial u}{\partial n} \right]_\Gamma &= Q, \\
u = g \quad \text{on } \partial \Omega. \tag{1.5}
\end{align*}
\]

Here, \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with its boundary \( \partial \Omega \). Both \( \Omega^- \) and \( \Omega^+ \) are subdomains of \( \Omega \) such that \( \Omega^- \cap \Omega^+ = \emptyset \) and \( \overline{\Omega^-} \cup \overline{\Omega^+} = \overline{\Omega} \), where an overline denotes the closure; see Figure 1.1 for an illustration. For simplicity, we assume that \( \overline{\Omega^-} \cap \partial \Omega = \emptyset \). We denote \( \Gamma = \overline{\Omega^-} \cap \overline{\Omega^+} \), and call it the interface separating \( \Omega^- \) and \( \Omega^+ \). We shall assume that \( \Gamma \) is sufficiently smooth. We also denote by \( n \) the unit vector normal to \( \Gamma \) pointing from \( \Omega^- \) to \( \Omega^+ \), or the unit exterior normal to \( \partial \Omega \); and by \( \partial/\partial n \) or \( \partial_n \) the corresponding normal derivative.

![Diagram](image)

Figure 1.1: A diagram of the geometry of an elliptic interface problem.

All the functions \( \beta, f : \Omega \to \mathbb{R}, w, Q : \Gamma \to \mathbb{R}, \) and \( g : \partial \Omega \to \mathbb{R} \) are given. As usual, for any function \( \xi : \Omega \to \mathbb{R} \), we denote the restrictions of \( \xi \) on \( \Omega^- \) and \( \Omega^+ \) by

\[
\xi^- = \xi|_{\Omega^-} \quad \text{and} \quad \xi^+ = \xi|_{\Omega^+},
\]

respectively; and denote by

\[
[\xi]_\Gamma (x) = \lim_{y \to x, y \in \Omega^+} \xi^+(y) - \lim_{y \to x, y \in \Omega^-} \xi^-(y),
\]
the jump of $\xi$ across the interface $\Gamma$ at $x \in \Gamma$, when the unique limiting values on $\Gamma$ of $\xi$ from both sides of $\Gamma$ exist. We assume that $\beta^-$ and $\beta^+$ are smooth and bounded on $\Omega^-$ and $\Omega^+$, respectively,

$$\beta(x) \geq \beta_0 \quad \forall x \in \Omega$$

for some constant $\beta_0 > 0$, $f \in L^2(\Omega)$, all $w$, $Q$, and $g$ are smooth and bounded.

Equations (1.3) and (1.4) are non-homogeneous interface or jump conditions. Elliptic interface problems (1.1)–(1.5) with such jump conditions arise in many areas. For example, in a Burton-Cabrera-Frank type model for epitaxial growth of thin films, the adatom (adsorbed adom) density that solves diffusion equation on terraces, and the corresponding flux, can both have jumps across interfaces that represent atomic steps [3, 5, 6]. Another example is that the reaction potential of electrostatics of a solvation energy satisfies a non-homogeneous jump condition for the flux [13].

If the jump conditions are homogeneous, i.e., $w = 0$ and $Q = 0$ on $\Gamma$ (cf. (1.3) and (1.4)), then the problem (1.1)–(1.5) is equivalent to that of finding $u \in H^1(\Omega)$ with $u = g$ on $\partial \Omega$ such that

$$\int_{\Omega} \beta \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1_0(\Omega).$$

In general, we can extend the non-homogeneous jump $w : \Gamma \to \mathbb{R}$ to a piecewise smooth function $\tilde{w} : \Omega \to \mathbb{R}$ such that

$$[\tilde{w}]_{\Gamma} = w \quad \text{and} \quad \tilde{w} = g \quad \text{on } \partial \Omega.$$

Then, the problem (1.1)–(1.5) is equivalent to the problem for $u = q + \tilde{w}$ with $q \in H^1_0(\Omega)$ uniquely determined by

$$\int_{\Omega} \beta \nabla q \cdot \nabla v \, dx = \int_{\Omega} f v \, dx - \int_{\Gamma} Q v \, ds - \int_{\Omega} \beta \nabla \tilde{w} \cdot \nabla v \, dx \quad \forall v \in H^1_0(\Omega).$$

In a standard, body-fitted finite-element method for solving the problem in the weak formulation (1.7), the interface is covered by a low dimensional finite-element mesh; and the extension of $w : \Gamma \to \mathbb{R}$ to $\tilde{w} : \Omega \to \mathbb{R}$ can be made by finite-element approximations.

In this work, we develop a class of finite-element methods based on non-body-fitted finite-element meshes to solve the elliptic interface problem (1.1)–(1.5). Such a method has the following distinguished features:

(a) A fixed finite-element mesh is used that allows the interface to cut through edges of elements.

(b) Both the jumps $w : \Gamma \to \mathbb{R}$ and $Q : \Gamma \to \mathbb{R}$ are extended locally by a single function using a level-set representation of the interface $\Gamma$. The extensions are necessary to transfer the original problem to a new one with homogeneous jump conditions so that the immersed finite element methods developed in [18] can be applied. The new methods make it possible to maintain second order accuracy on the elements near the interface, see Section 2 and Section 5.
(c) Special finite-element basis functions for nodal points near the interface are constructed to satisfy the homogeneous jump conditions, see Section 4.

(d) The resulting linear system of discretization is symmetric positive definite.

(e) Optimal convergence rates, same as those for a body-fitted finite-element method, are achieved, see Section 3 for basic error estimates and Section 6 for numerical results on such convergence properties.

Our methods incorporate the finite-element discretization into the framework of immersed-interface method for interface problems; and we therefore call them immersed-interface finite-element methods. The basic idea of an immersed-interface method is to incorporate the jump conditions in constructing basis functions. In a finite-difference immersed-interface method, the jump conditions are enforced through finite-difference equations on grid points near the interface. In our immersed-interface finite-element methods, the jump conditions are enforced through the construction of special finite-element basis functions that satisfy the homogeneous interface conditions. Clearly, such basis functions depend on the interface location and the jump \( w \). Some of the related work can be found in [7, 12].

The development of our methods is strongly motivated by the need of a fast and accurate solver of elliptic interface problems that is required in each time step in a long-time level-set simulation of interface dynamics without re-meshing. Such simulation has been a powerful numerical approach in understanding material properties, biological processes, and many other important phenomena in science and engineering [20–22].

The finite element spaces constructed in this paper are conforming ones that contain piecewise \( P_1 \) polynomials. The interpolation error is therefore of second order. If the solution to the elliptic interface problem is piecewise smooth, or more precisely, \( u^\pm \in C^2(\Omega^\pm) \), which is true for many applications, then the standard convergence results are true for our finite element methods. This means that our finite-element solution is of first order accurate in the \( H^1 \) norm, and second order accurate in the \( L^2 \) norm. In this work, we only construct some two-dimensional, linear and quadratic, triangular, immersed-interface finite elements. However, our framework described here can be used for high-order, immersed-interface finite elements and for more general elliptic interface problems in two and three space dimensions.

We use the zero level set of a Lipschitz continuous function, particularly, a good approximation of the signed distance function, to represent the interface \( \Gamma \). For such a level set function, the interface can only cut the interface once between two grid lines. The resolution of the interface \( \Gamma \) is determined by the level set function. We note that there are plenty of discussions in the literature about how to construct the level set function for an interface \( \Gamma \), see, e.g., [20, 22]. If the interface is complicated in reference to a mesh, then it may not be resolved well by the level set function. Consequently, the finite-element solution obtained from our method may not resolve all the fine details of the solution. A finer mesh is then needed to resolve the geometry and the solution.
Our work combines the development of new formulation of underlying problems, construction of new elements, basic error estimates, and numerical experiments. The analysis in our work focuses on two parts: one is the algorithm of removing source singularities, and the other is the interpolation error estimate for some of the immersed-interface finite element approximations. We have not tried to put in this single paper the analysis of other errors, such as those of the approximation of the interface \( \Gamma \) by \( \Gamma_h \), numerical quadrature, round-off errors, etc.

In Section 2, we present a new weak formulation of the elliptic interface problem (1.1)-(1.5) with a special treatment of non-homogeneous jump conditions. In Section 3, we describe the framework of our immersed-interface finite-element methods, and provide basic error estimates. In Section 4, we construct several linear or quadratic, immersed-interface finite-element spaces and obtain their corresponding interpolation errors. In Section 5, we give some details of implementation of our methods. In Section 6, we present numerical results. Finally, in Section 7, we draw conclusions.

## 2 Weak Formulation

We first review the level-set representation of the interface \( \Gamma \) which is assumed to be smooth. Let \( \varphi : \Omega \to \mathbb{R} \) be a continuous function that satisfies

\[
\varphi(x) = \begin{cases} 
< 0 & \text{if } x \in \Omega^-, \\
= 0 & \text{if } x \in \Gamma, \\
> 0 & \text{if } x \in \Omega^+. 
\end{cases} 
\]

We call such a function \( \varphi : \Omega \to \mathbb{R} \) a level-set function that represents the interface \( \Gamma \). For \( \rho > 0 \), we denote the \( \rho \)-neighborhood of \( \Gamma \) in \( \Omega \) by

\[
N(\Gamma, \rho) = \{ x \in \Omega : \text{dist}(x, \Gamma) < \rho \},
\]

where \( \text{dist}(x, \Gamma) \) is the distance from \( x \) to \( \Gamma \). We assume that there exists \( \rho_0 > 0 \) such that \( N(\Gamma, \rho_0) \subset \Omega \), and

\[
\varphi \text{ is smooth and } |\nabla \varphi| > 0 \text{ in } N(\Gamma, \rho_0).
\]

We note that the unit normal \( n \) to \( \Gamma \), pointing from \( \Omega^- \) to \( \Omega^+ \), is given by

\[
n = \frac{\nabla \varphi}{|\nabla \varphi|}.
\]

The signed distance function

\[
\varphi(x) = \begin{cases} 
- \text{dist}(x, \Gamma) & \text{if } x \in \Omega^-, \\
0 & \text{if } x \in \Gamma, \\
+ \text{dist}(x, \Gamma) & \text{if } x \in \Omega^+.
\end{cases} 
\]
is a typical level-set function that satisfies our assumptions.

We now turn to the description of our treatment of non-homogeneous jump conditions. We first need the following lemma:

**Lemma 2.1** Let \( \rho > 0 \) be small enough. Then, for any \( x \in N(\Gamma, \rho) \), there exists a unique \( \hat{x} \in \Gamma \) such that

\[
|x - \hat{x}| = \text{dist}(x, \Gamma).
\]

Moreover,

\[
\frac{x - \hat{x}}{|x - \hat{x}|} = \begin{cases} 
- n(\hat{x}) & \text{if } x \in \Omega^-, \\
+ n(\hat{x}) & \text{if } x \in \Omega^+,
\end{cases}
\]

where \( n(\hat{x}) \) is the unit normal to \( \Gamma \) at \( \hat{x} \), pointing from \( \Omega^- \) to \( \Omega^+ \).

**Proof.** Without loss of generality, let us assume that, in a local Cartesian coordinate system, \( x = (0, 0) \) is the origin, and the interface nearby is a graph of a smooth function \( \eta = \eta(s) \) that is nonzero for any \( s \) in a certain range. The distance from the origin to any point \( (s, \eta(s)) \) on the interface is then given by \( \sqrt{I(s)} < \rho \) with

\[
I(s) = s^2 + |\eta(s)|^2.
\]

Setting \( I'(s) = 0 \), we get \( \eta'(s) = -s/\eta(s) \). This implies (2.5), since \( (-1, \eta'(s)) \) is parallel to the normal at \( (s, \eta(s)) \). Moreover,

\[
I''(s) = 2 + 2|\eta'(s)|^2 + 2\eta(s)\eta''(s)
\]

is positive, since \( I(s) \), and hence \( \eta(s) \), is small enough, whenever \( \rho > 0 \) is small enough. Thus, there exists a unique \( s \) that minimizes \( I(s) \). This implies the existence and uniqueness of \( \hat{x} \) that satisfies (2.4). \( \text{Q.E.D.} \)

In what follows, we fix \( \rho \in \mathbb{R} \) as in the above lemma and assume that \( 0 < \rho < \rho_0 \), where \( \rho_0 \) is the same as in (2.2). We define \( w_\rho : \Omega \to \mathbb{R} \) and \( Q_\rho : \Omega \to \mathbb{R} \) to be the extension of \( w : \Gamma \to \mathbb{R} \) and \( Q : \Gamma \to \mathbb{R} \), respectively, that satisfy the following:

**E1.** Both \( w_\rho \) and \( Q_\rho \) are smooth on \( \overline{\Omega} \);

**E2.** \( w_\rho(x) = w(\hat{x}) \) and \( Q_\rho(x) = Q(\hat{x}) \) for any \( x \in N(\Gamma, \rho) \), where \( \hat{x} \) is defined as in Lemma 2.1;

**E3.** \( w_\rho(x) = g(x) \) for any \( x \in \partial\Omega \), and \( Q_\rho(x) = 0 \) for any \( x \in \Omega \setminus N(\Gamma, \rho) \).

To see the existence of \( w_\rho \), we first apply Lemma 2.1 to define a function, say \( w_1 \), by \( w_1(x) = w(\hat{x}) \) for any \( x \in N(\Gamma, 2\rho) \) for \( \rho > 0 \) small enough, and \( w_1 = 0 \) elsewhere in \( \Omega \). Using the lifting operator \([2,11]\), there exists a smooth function, say \( w_2 \), on \( \Omega \) such that the restriction of \( w_2 \) on \( \partial\Omega \) is \( g \). (The smoothness depends on that of \( \partial\Omega \).) Now, assume
that \( N(\Gamma, 2\rho) \) and \( A := N(\partial \Omega, \varepsilon) = \{ x \in \Omega : \text{dist} (x, \partial \Omega) < \varepsilon \} \) are disjoint for some small \( \varepsilon > 0 \). Then, by applying mollifiers to \( w_1 + \chi_A w_2 \), we obtain the desired \( w_\rho \). The existence of \( Q_\rho \) is seen similarly.

We define \( u_\rho : \Omega \to \mathbb{R} \) by

\[
u_\rho(x) = \chi_{\Omega^+}(x) \left( w_\rho(x) + \frac{Q_\rho(x)}{\beta^+(x) |\nabla \varphi(x)|} \right) \quad \forall x \in \Omega, \quad (2.6)
\]

where \( \chi_{\Omega^+} \) is the characteristic function of \( \Omega^+ \). Note by \textbf{E3} that \( Q_\rho = 0 \) outside \( N(\Gamma, \rho_0) \) in which \( \varphi \) may not be smooth. Thus, implicitly, the second term in \( u_\rho \) is defined to be 0 outside \( N(\Gamma, \rho_0) \).

The following statement summarizes some useful properties of \( u_\rho : \Omega \to \mathbb{R} \):

\textbf{Lemma 2.2} Both \( u^-_\rho \) and \( u^+_\rho \) are smooth on \( \Omega^- \) and \( \Omega^+ \), respectively. Moreover, \( u_\rho = g \) on \( \partial \Omega \), and

\[
[u_\rho]_\Gamma = w, \quad \left[ \beta \frac{\partial u_\rho}{\partial n} \right]_\Gamma = Q.
\]

\textbf{Proof}. Since \( \beta^+ \) is smooth and bounded on \( \Omega^+ \), it follows from (1.6), \textbf{E1}, and (2.2) that \( u^+_\rho \) is smooth on \( \Omega^+ \). Obviously, \( u^-_\rho = 0 \) is smooth on \( \Omega^- \). By \textbf{E3}, \( u_\rho = g \) on \( \partial \Omega \). By (2.1) and \textbf{E2},

\[
[u_\rho]_\Gamma = u^+_\rho - u^-_\rho = w.
\]

Now, fix \( x \in \Gamma \). It follows from Lemma 2.1 and \textbf{E2} that \( \partial_n w_\rho = 0 \). Moreover, \( \varphi(x) = 0 \) by (2.1). Therefore, for any \( x \in \Gamma \), we obtain by \textbf{E2}, (2.6), (2.3), and the fact that \( \varphi(x) = 0 \) that

\[
\left[ \beta(x) \frac{\partial u_\rho(x)}{\partial n} \right]_\Gamma = \beta^+(x) \frac{\partial u^+_\rho(x)}{\partial n}
\]

\[
= \beta^+(x) \frac{\partial w_\rho(x)}{\partial n} + \beta^+(x) \frac{\partial}{\partial n} \left( \frac{Q_\rho(x)}{\beta^+(x) |\nabla \varphi(x)|} \right) \varphi(x) + \frac{Q_\rho(x)}{|\nabla \varphi(x)|} \frac{\partial \varphi(x)}{\partial n}
\]

\[
= Q(x) \frac{n(x) \cdot \nabla \varphi(x)}{|\nabla \varphi(x)|} = Q(x).
\]

The proof is completed. \textbf{Q.E.D.}

\textbf{Theorem 2.1} There exists a unique \( q \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} \beta \nabla q \cdot \nabla v \, dx = \int_{\Omega} f v \, dx - \int_{\Gamma} Q v \, ds - \int_{\Omega^+} \beta^+ \nabla u_\rho \cdot \nabla v \, dx \quad \forall v \in H^1_0(\Omega). \quad (2.7)
\]
This is equivalent to
\[
\int_{\Omega} \beta \nabla q \cdot \nabla v \, dx = \int_{\Omega} f v \, dx + \int_{\Omega^+} (\nabla \cdot (\beta^+ \nabla u_\rho)) v \, dx \quad \forall v \in H^1_0(\Omega). \tag{2.8}
\]
Moreover, \( u = q + u_\rho \) solves the problem (1.1)-(1.5). In particular, \( q \in H^1_0(\Omega) \) satisfies the homogeneous jump conditions
\[
[q]_\Gamma = 0 \quad \text{and} \quad \left[ \beta \frac{\partial q}{\partial n} \right]_\Gamma = 0. \tag{2.9}
\]

**Proof.** Note that the line integral in (2.7) defines a continuous, linear functional on \( H^1_0(\Omega) \). The existence and uniqueness of \( q \in H^1_0(\Omega) \) that satisfies (2.7) follows therefore from the Lax-Milgram Theorem. By integration by parts, (2.6), and Lemma 2.2, we imply that (2.7) and (2.8) are equivalent.

Choosing \( v \in H^1_0(\Omega) \) with \( \text{supp} \, v \subset \Omega^- \) and \( \text{supp} \, v \subset \Omega^+ \), respectively, applying the regularity theory of elliptic problems, we obtain from (2.7) that \( u = q + u_\rho \) satisfies (1.1) and (1.2). The jump condition (1.3) follows from Lemma 2.2. Now, integrating by parts, we obtain by (1.1), (1.2), (2.6), and (2.7) that
\[
\int_{\Gamma} \left( \left[ \beta \frac{\partial u}{\partial n} \right]_\Gamma - Q \right) v \, ds = 0 \quad \forall v \in H^1_0(\Omega).
\]
This leads to (1.4). Clearly, the function \( u = q + u_\rho \) satisfies (1.5).

Finally, since \( q = u - u_\rho \), we have
\[
[q]_\Gamma = [u]_\Gamma - [u_\rho]_\Gamma = w - w = 0
\]
from (1.3) and Lemma 2.2. We also have
\[
\left[ \beta \frac{\partial q}{\partial n} \right]_\Gamma = \left[ \beta \frac{\partial u}{\partial n} \right]_\Gamma - \left[ \beta \frac{\partial u_\rho}{\partial n} \right]_\Gamma = Q - Q = 0
\]
from (1.4) and Lemma 2.2. **Q.E.D.**

### 3 Immersed-Interface Finite-Element Approximations

In this section, we first state practically reasonable assumptions on how the interface can cut edges of elements in a finite-element mesh. We then define immersed-interface finite-element approximations of our underlying problem formulated in Theorem 2.1. Finally, we give rigorous error estimates for our method.

Let \( \mathcal{T}_h \) be a finite-element mesh with mesh size \( h \) that covers \( \overline{\Omega} \). We assume that the elements in \( \mathcal{T}_h \) are all triangles. For simplicity, we shall assume that \( \Omega \) is a convex
polygonal domain and the mesh covers $\Omega$ exactly. Standard finite-element techniques can be applied to treat a curved boundary without affecting our approximation properties. We remark that in practice the computational domain $\Omega$ can often be chosen as a rectangular domain with sides parallel to the coordinate axes; and the finite-element mesh can be uniform.

We call an element $T \in \mathcal{T}_h$ an interface element, if $\Gamma \cap \text{int} T \neq \emptyset$. Note that an element is a non-interface element, if one of its edges is part of the interface. We assume for any interface element $T \in \mathcal{T}_h$, the set $\Gamma \cap \partial T$ consists of exactly two points that are on different edges of $T$.

We define an immersed-interface finite-element space $V_h$ with respect to the mesh $\mathcal{T}_h$ to be a finitely-dimensional subspace of $L^2(\Omega)$ that consists of all the linear combinations of the corresponding basis functions $\phi_1, \ldots, \phi_N$ for some integer $N \geq 1$:

$$V_h = \text{span} \{ \phi_1, \ldots, \phi_N \}. \quad (3.1)$$

The basis functions are the usual finite-element basis functions on a non-interface element, and are piecewise polynomials on each interface element that is determined by the element and the interface $\Gamma$. All the basis functions satisfy the homogeneous jump conditions for both of the function and flux. Moreover, there exists an interpolation operator from some functional space to $V_h$ that enjoys the usual approximation properties.

To be more precise, we define a conforming, immersed-interface finite-element space $V_h$ for the approximation of a second-order elliptic interface problem to be a subspace of $H^1(\Omega)$ that satisfies the following properties:

**FE1.** All the functions in $V_h$ restricted onto the union of all non-interface elements form a usual finite-element space of piecewise polynomials of degree $\leq k$ for some integer $k \geq 1$, such as a $P_k$ type Lagrange finite-element space when $\mathcal{T}_h$ is a triangular mesh, or a $Q_k$ or $Q'_k$ type Lagrange finite-element space when $\mathcal{T}_h$ is a quadrilateral mesh [4,8]. On each interface element $T \in \mathcal{T}_h$, all functions in $V_h$ are piecewise polynomials. The support of each basis function is a union of a few elements;

**FE2.** All the basis functions, and hence all the functions in $V_h$, satisfy the homogeneous jump conditions

$$[v_h]_{\Gamma_h} = 0 \quad \text{and} \quad \left[ \beta \frac{\partial v_h}{\partial n} \right]_{\Gamma_h} = 0 \quad \forall v_h \in V_h, \quad (3.2)$$

where $\Gamma_h$ denotes the union of line segments approximating the interface $\Gamma$, and the normal derivatives of $v_h \in V_h$ from each side of the interface are assumed to exist;

**FE3.** There exists an interpolation operator $I_h : D(\Omega) \to V_h$ such that

$$\| I_h v - v \|_{H^1(\Omega)} \leq C h^{p(k)} \| v \|_{H^{p+1}(\Omega \setminus \Gamma)} \quad \forall v \in D(\Omega), \quad (3.3)$$

9
where $D(\Omega) \subset H^1(\Omega)$ is an infinite-dimensional space of functions that are smooth and bounded on both $\Omega^-$ and $\Omega^+$, respectively, $\tilde{k}$ is a constant such that $0 < \sigma(k) \leq k$, and $C > 0$ denotes a generic constant independent of $h$ and $v$.

The homogeneous jump condition for $v_h \in V_h$ in (3.2) is a consequence of the fact that $V_h \subset H^1(\Omega)$. Often the interpolation error in FE3 can be the same as usual, i.e., $\tilde{k} = k$. But, it can be slightly worse due to the approximation of $\Gamma$ by $\Gamma_h$.

We can define similarly a nonconforming, immersed-interface finite-element space $V_h \subset L^\infty(\Omega)$ with $V_h \not\subset H^1(\Omega)$. For a nonconforming immersed-interface finite-element space $V_h$ that is used to solve our underlying problem with the boundary condition (1.5), we also need to assume that a discrete Poincaré inequality,

$$
\|v_h\|_{L^2(\Omega)}^2 \leq C \sum_{T \in T_h} \|
abla v_h\|_{L^2(T \setminus \Gamma_h)}^2, \quad \forall v_h \in V_h,
$$

is satisfied. Several examples of conforming and nonconforming, linear or quadratic, immersed-interface finite elements will be presented in the next section.

We consider now immersed-interface finite-element approximations of the solution $q \in H^1_0(\Omega)$ defined by (2.7). Let $u_\rho : \Omega \to \mathbb{R}$ be given in (2.6) for some $\rho > 0$ small enough. Let now $V_h \subset H^1_0(\Omega)$ be a conforming, immersed-interface finite-element space of piecewise polynomials of degree $\leq k$ as in FE1-FE3 that accounts for the boundary condition (1.5). To be precise, we assume in our analysis below that the domain of the interpolation operator $I_h$ is

$$
D(\Omega) = \{ v \in H^1_0(\Omega) : v^- \in H^{k+1}(\Omega^-) \cap C(\Omega^-) \text{ and } v^+ \in H^{k+1}(\Omega^+) \cap C(\Omega^+) \}.
$$

**Theorem 3.1** There exists a unique $q_h \in V_h$ such that

$$
\int_\Omega \beta \nabla q_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx + \int_{\Omega^+} (\nabla \cdot \beta^+ \nabla u_\rho) v_h \, dx \quad \forall v \in V_h. \tag{3.4}
$$

Moreover, if the solution $q \in H^1_0(\Omega)$ of (2.7) satisfies that $q^- \in H^{k+1}(\Omega^-) \cap C(\overline{\Omega^-})$ and $q^+ \in H^{k+1}(\Omega^+) \cap C(\overline{\Omega^+})$, then

$$
\|q - q_h\|_{H^1(\Omega)} \leq Ch^{\sigma(k)} \|q\|_{H^{k+1}(\Omega \setminus \Gamma)}, \tag{3.5}
$$

$$
\|q - q_h\|_{L^2(\Omega)} \leq Ch^{\sigma(k)+\sigma(1)} \|q\|_{H^{k+1}(\Omega \setminus \Gamma)}, \tag{3.6}
$$

where $\sigma(k)$ is defined in FE3 and $C > 0$ is a constant independent of $h$ and $q$.

**Proof.** Since $V_h \subset H^1_0(\Omega)$, the existence and uniqueness of $q_h \in V_h$ that satisfies (3.4) follows from the Lax-Milgram Theorem. Moreover, by (2.8), which is equivalent to (2.7) by Theorem 2.1, and (3.4), we obtain

$$
\int_\Omega \beta \nabla (q - q_h) \cdot \nabla v_h \, dx = 0 \quad \forall v_h \in V_h. \tag{3.7}
$$
The error estimate (3.5) can be obtained by a standard argument using a Poincaré inequality, the ellipticity indicated by (1.6), the error equation (3.7), and the interpolation error estimate (3.3), see [4,8].

To obtain the $L^2$ error estimate (3.6), we use the standard dual argument (cf. e.g., [4,8]) with slight modification taking care of the lack of global regularity of a solution. Let $r \in H^1_0(\Omega)$ be the unique function satisfies

$$
\int_\Omega \beta \nabla r \cdot \nabla v \, dx = \int_\Omega (q - q_h)v \, dx \quad \forall v \in H^1_0(\Omega). \tag{3.8}
$$

Since $\beta^-$ and $\beta^+$ are smooth and bounded in $\Omega^-$ and $\Omega^+$, respectively, we see that $r^- \in H^2(\Omega)$ and $r^+ \in H^2(\Omega)$. Moreover, it follows from (3.8) that $[\beta \partial_n r]_\Gamma = 0$. Therefore, we have (cf. Theorem 2.1 in [7]) that

$$
\|r^\pm\|_{H^2(\Omega^\pm)} \leq C\|q - q_h\|_{L^2(\Omega)}. \tag{3.9}
$$

Setting $v = q - q_h \in H^1_0(\Omega)$ in (3.8), we have by (3.8), (3.7) with $v_n = I_h r$, the Cauchy-Schwartz inequality, (3.3) with $k = 1$, (3.5), and (3.9) that

$$
\|q - q_h\|_{L^2(\Omega)}^2 = \int_\Omega \beta \nabla r \cdot \nabla (q - q_h) \, dx
= \int_\Omega \beta \nabla (r - I_h r) \cdot \nabla (q - q_h) \, dx
\leq C\|r - I_h r\|_{H^1(\Omega)}\|q - q_h\|_{H^1(\Omega)}
\leq C h^{\sigma(k) + \sigma(1)}\|r\|_{H^2(\Omega \backslash \Gamma)}\|q\|_{H^{k+1}(\Omega \backslash \Gamma)}
\leq C h^{\sigma(k) + \sigma(1)}\|q\|_{H^{k+1}(\Omega \backslash \Gamma)}\|q - q_h\|_{L^2(\Omega)},
$$

leading to (3.6). \textbf{Q.E.D.}

We remark that the removal of both jumps $w$ and $Q$ using a single function leads to finite-element equation (3.4) that does not involve line integrals as in (2.7). This much simplifies numerical implementation.

\section{Construction of Basis Functions}

We now describe how to construct some new classes of triangular, immersed-interface finite-element basis functions that satisfy the homogeneous jump conditions.

Let $z_1, \ldots, z_N$ be all the vertices of triangular elements in the mesh $\mathcal{T}_h$. We define our linear or quadratic linear immersed-interface finite-element basis functions $\phi_i$ associated with $z_i$ ($i = 1, \ldots, N$) to be functions on $\Omega$ that satisfy the following properties:
B1. Each $\phi_i$ ($1 \leq i \leq N$) is linear on a non-interface element and is piecewise linear or quadratic on an interface element;

B2. $\phi_i(z_j) = \delta_{ij}$ for $i, j = 1, \ldots, N$, where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$;

B3. All $\phi_i$ ($1 \leq i \leq N$) satisfy the approximate homogeneous jump conditions:

$$[\phi_i]_{\Gamma_h} = 0, \quad \left[ \beta \frac{\partial \phi_i}{\partial n} \right]_{\Gamma_h} = 0.$$ 

The corresponding finite-element space $V_h$ is given by (3.1).

We call a vertex an irregular vertex or irregular node, if it is a vertex of an interface element; and a regular vertex or regular node, otherwise. For a regular node $z_i$, we define $\phi_i$ to be the usual conforming, linear finite-element basis function associated with $z_i$, i.e., $\phi_i \in C(\bar{\Omega}) \cap H^1_0(\Omega)$, $\phi_i$ is a linear polynomial on each element $T \in \mathcal{T}_h$, and $\phi_i(z_j) = \delta_{ij}$ for $j = 1, \ldots, N$.

For an irregular node $z_i$, we construct the corresponding basis function by modifying the usual linear finite-element basis function to satisfy the properties B1–B3. We need only to construct a local basis (or shape) function on an interface element for an irregular node.

4.1 A nonconforming linear element

This nonconforming, immersed-interface finite-element was first constructed in [18]; see also [12]. We briefly review it, since it will be used to construct our new linear and quadratic elements.

Fix an interface element $T \in \mathcal{T}_h$, cf. Figure 4.1. As in the common practice, we approximate the interface in $T$, $\Gamma \cap T$, by a line segment connecting the intersections of the interface and the edges of the triangle $T$. This line segment is $\overline{DE}$ in Figure 4.1. The line segment divides $T$ into two parts $T^+$ and $T^-$: one triangular and the other quadrilateral. Note that there is a small region $T_r$ in $T$,

$$T_r = T - \Omega^+ \cap T^+ - \Omega^- \cap T^-,$$

whose area is of order $O(h^3)$.

We now consider the element $T$ in Figure 4.1 as a reference interface element $T$ and define local basis function for each vertex of $T$. The local basis function for a general interface element in the mesh $\mathcal{T}_h$ can be defined through the usual affine transformation. We denote

$$T = \{(x_1, x_2) : 0 \leq x_1 \leq h, \quad 0 \leq x_2 \leq h, \quad x_1 + x_2 \leq h\},$$

and assume that the coordinates at $A$, $B$, $C$, $D$, and $E$ are

$$(0, h), \quad (0, 0), \quad (h, 0), \quad (0, y_1), \quad (h - y_2, y_2),$$

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Figure 4.1: A typical interface element $T = \triangle ABC$. The arc $DME$ is the part of the interface $\Gamma$ in $T$. It is approximated by the line segment $DE$. $T^+ = \triangle ADE$, $T^- = T - T^+$, and $T_r$ is the region enclosed by the $DE$ and the arc $DME$.

respectively; cf. Figure 4.1.

Each of the three local basis functions corresponding to the nodes $A$, $B$, or $C$ takes value 1 at one node and 0 at the other two. Once the values at nodes $A$, $B$, and $C$ are specified, a local, nonconforming finite-element basis function $\phi$ for this interface triangle is determined by

$$
\phi(x) = \begin{cases} 
\phi^+(x) = a_0 + a_1 x_1 + a_2(x_2 - h) & \text{if } x = (x_1, x_2) \in T^+, \\
\phi^-(x) = b_0 + b_1 x_1 + b_2 x_2 & \text{if } x = (x_1, x_2) \in T^-.
\end{cases}
$$  \hspace{1cm} (4.1)

The coefficients $a_i$ and $b_i$ ($i = 0, 1, 2$) are determined by the conditions (cf. [18]).

$$
\phi^+(D) = \phi^-(D), \quad \phi^+(E) = \phi^-(E), \quad \beta^+ \frac{\partial \phi^+}{\partial n} = \beta^- \frac{\partial \phi^-}{\partial n},
$$  \hspace{1cm} (4.2)

where $n$ is the unit normal direction of the line segment $DE$. It is shown in [18] that this function $\phi$ is uniquely determined. Note that basis functions defined in this way can be discontinuous across edges of interface elements. So, this defines a nonconforming finite element.

4.2 A conforming linear element

This conforming element was proposed in [18]. Both error estimates and numerical experiments in [18] show that the corresponding interpolation errors that can depend on an angle condition are of second order which is optimal [18].

Let $T = \triangle ABC$ be an interface element; cf. Figure 4.2. As before, we assume that the interface meets edges of this element at $D$ and $E$. To construct a local basis function that is globally continuous, we extend the previously defined basis function at the same
node (vertex) to one more triangle along the interface (cf. Figure 4.2 (b)). We require the local basis functions in two adjacent interface elements, such as $\Delta ABC$ and $\Delta AFB$, take the same value at the interface point on their common edge, such as the point $D$. This will achieve the global continuity of a basis function associated with an irregular node.

We construct a local basis function $\phi$ by assigning its values at the vertices $A, B, C, F,$ and $I$, respectively. This construction consists of the following five steps:

**P1.** Use the values at the nodes $A, B, C, F,$ and $I$ to construct the three nonconforming finite-element basis functions defined as in Subsection 4.1 on the elements $\Delta ABC$, $\Delta AFB$, and $\Delta ACI$, respectively;

**P2.** Set the value at $D$ as the average of the values at $D$ of the nonconforming, piecewise linear, basis functions defined on $\Delta ABC$ and $\Delta AFB$ constructed in **P1**;

**P3.** Similarly, set the value at $E$ as the average of values at $E$ of the nonconforming, piecewise linear, basis functions defined on the elements $\Delta ABC$ and $\Delta ACI$ constructed in **P1**;

**P4.** Partition the element $\Delta ABC$ into three sub-triangles by an auxiliary line, say line segment $BE$, or $DC$. We choose the auxiliary line in such a way that at least one of angles (or complimentary angle if the angle is more than $\pi/4$) is bigger than or equal to $\pi/2$;

**P5.** Define the basis function to be the piecewise linear function in the three sub-triangles determined by the values at the points $A, B, C, D$ and $E$.

For this type of conforming finite element space and homogeneous jump conditions, i.e., $w = 0$ and $Q = 0$, we have the following error estimates of the interpolation function

\[
\|I_h u - u\|_\infty \leq Ch^2\|D^2u\|_{\infty, \Omega \setminus (\cup T_r)}, \quad (4.3)
\]

\[
\|I_h u - u\|_{H^1(\Omega)} \leq Ch\|D^2u\|_{\infty, \Omega \setminus (\cup T_r)}, \quad (4.4)
\]
assuming that the solution is piecewise smooth in $\Omega^-$ and $\Omega^+$, where $C > 0$ is a generic constant independent of $h$ and $\cup T_e$ is the region of mis-matched region between $\Gamma$ and $\Gamma_h$, see [18] for the proof.

### 4.3 A conforming quadratic element

We now construct a conforming quadratic element. As before, the basis functions associated with regular nodes are the standard conforming linear finite-element basis functions. But, the basis functions associated with irregular nodes are piecewise quadratic. All the basis functions are globally continuous. The idea is to average the tangential derivatives in the construction in Subsection 4.2. Referring to Figure 4.2, we describe this procedure as follows:

**P1.** Use the values at the nodes $A$, $B$, $C$, $F$, and $I$ to construct the three nonconforming linear finite-element basis functions defined in Subsection 4.1 on the elements $\Delta ABC$, $\Delta AFB$, and $\Delta ACI$, respectively;

**P2.** Set the value at $D$ as the average of the values at $D$ of the nonconforming linear finite-element basis functions defined $\Delta ABC$ and $\Delta AFB$ constructed in P1. Assign the tangential derivative at $D$ along $AD$ to be the average of the values of the tangential derivatives (along $AD$) of the nonconforming linear finite-element basis functions on $\Delta ABC$ and $\Delta AFB$ constructed in P1. Similarly, assign the tangential derivative at $D$ along $DB$ as the average of the values of the tangential derivatives (along $DB$) of the nonconforming finite-element basis functions on $\Delta ABC$ and $\Delta AFB$ constructed in P1;

**P3.** Repeat to set the value at $E$ as the average of values at $E$ of the nonconforming linear finite-element basis functions defined on the elements $\Delta ABC$ and $\Delta ACI$ in P1. Also assign the tangential derivatives along $AE$ and $EC$ as the average of those from the nonconforming finite-element basis functions along the same edge;

**P4.** Partition the element $\Delta ABC$ into three sub-triangles by an auxiliary line. We choose the auxiliary line in such a way that at least one of the angles (or complimentary angle if the angle is bigger than $\pi/4$) is bigger than or equal to $\pi/2$. In Figure 4.2, for example, the angles are $\angle EDB$, $\angle DEB$, and $\angle EBD$, where the complimentary angle of $\angle EDB$ is bigger than $\pi/4$;

**P5.** Assign the tangential derivatives along $DE$ and $BE$ exactly the same (no average) as those from the nonconforming finite-element basis functions;

**P6.** Set the values of the tangential derivatives along $BE$ and $BC$ exact the same as those from the nonconforming finite-element basis function on $\Delta ABC$;
**P7.** Define the basis function $\psi_i$ to be the piecewise quadratic function in the three sub-triangles determined by the values at the points $A, B, C, D,$ and $E$, respectively, and the tangential derivatives from each side of the triangles; see Figure 4.3 for an illustration.

![Diagram of three triangles](image)

Figure 4.3: A diagram of three triangles on which piecewise continuous quadratic functions can be determined. The symbol $\rightarrow$ indicates a tangential derivative.

We now give an error estimate for the interpolation error of this element for a given piecewise smooth function $v$. We first show that a quadratic function on a triangle is uniquely determined by its values at the three vertices and its tangential derivatives along the three sides at some particular points.

**Lemma 4.1** Referring to the geometry in Fig. 4.4, there is a unique quadratic polynomial

$$I_h v(x, y) = v_A + a_{10} x + a_{01} y + a_{20} x^2 + a_{11} x y + a_{02} y^2. \quad (4.5)$$

that interpolates the values of $v_A$, $v_D$, and $v_E$, and three tangential derivatives $v_{AE}', v_{AD}',$ and $v_{ED}'$ defined at the three vertices $A$, $E$, and $D$ respectively.

**Proof:** Using the undetermined coefficient method, we find the unique expression of the coefficients$^1$

$$a_{10} = \frac{\tilde{h} v_{AD}' + 2v_D - 2v_A}{\tilde{h}}$$

$$a_{01} = -\frac{\tilde{h} v_{AD}' - \sqrt{2} v_{AE}' \tilde{h} + 2v_D - 2v_A}{\tilde{h}}$$

$$a_{20} = -\frac{\tilde{h} v_{AD}' + v_D - v_A}{\tilde{h}^2}$$

$^1$We use Maple to get the coefficients.
\[
a_{11} = \frac{-2\tilde{h}\alpha v'_D - 2\alpha v_D + 2\alpha v_A + \tilde{h}^2 v'_{AD} + 2\tilde{h} v_D}{\tilde{h}^2 \alpha} - \frac{\sqrt{\tilde{h}^2 - 2\tilde{h} \alpha + 2\alpha^2 v'_{ED} \tilde{h} + \tilde{h} \alpha \sqrt{2} v'_{AE} - 2\tilde{h} v_E}}{\tilde{h}^2 \alpha}
\]

\[
a_{02} = \frac{\tilde{h}^2 v'_{AD} \alpha + 2\tilde{h} \alpha v_D - \tilde{h}^2 \alpha \sqrt{2} v'_{AE} - \tilde{h} v'_{AD} \alpha^2 - \alpha^2 v_D + \alpha^2 v_A}{\tilde{h}^2 \alpha^2} + \frac{v_E \tilde{h}^2 - v_A \tilde{h}^2 + \sqrt{\tilde{h}^2 - 2\tilde{h} \alpha + 2\alpha^2 v'_{ED} \tilde{h} \alpha + \tilde{h} \alpha^2 \sqrt{2} v'_{AE} - 2v_E \tilde{h} \alpha}}{\tilde{h}^2 \alpha^2},
\]

where \( \tilde{h} \) is the distance from \( A \) to \( D \), and the coordinates at \( E \) is \((\alpha, \alpha)\). \textbf{Q.E.D.}

![Diagram](image)

**Figure 4.4:** A diagram in determining of the quadratic interpolation function.

There are two steps in obtaining the interpolation error. First, the values of the quadratic function according to our algorithm at vertices are either directly copied from the conforming linear basis function (at the vertices of the right triangles), or the average of values of the conforming linear basis function from the two adjacent triangles. Therefore those values are \( O(h^2) \) perturbations to \( v \) from the reference [18]. Similarly, the values of the tangential derivatives are \( O(h) \) perturbation to that of \( v \). Second, from the above reasoning, we can conclude that the values at the mid-points of each side of the triangle is \( O(h^2) \) perturbation to \( v \). Since the quadratic function can also be uniquely determined from its values at vertices and the three mid-points and it is insensitive to perturbations of those values, see [14], we conclude the interpolation function is \( O(h^2) \) perturbation to \( v(x) \) in the entire triangle.

The second part of the argument is a direct consequence of the interpolation error estimate for the standard conforming \( P_2 \) finite element. The first part of the argument is a consequence of the following result:

**Lemma 4.2** Let \( a, b \in \mathbb{R} \) with \( h := b - a > 0 \). Let \( v \in C^1[a, b] \). Then, there exists a unique quadratic polynomial \( p \) such that \( p(a) = v(a), \ p(b) = v(b), \) and \( p'(a) = v'(a) \).
Moreover, if \( v \) is smooth then
\[
\|v - p\|_{L^\infty}(a, b) \leq C h^3,
\]
\[
\|v' - p'\|_{L^\infty}(a, b) \leq C h^2,
\]
where \( C > 0 \) is a constant independent of \( h \).

**Proof.** Note that any quadratical polynomial \( q \) can be written as
\[
q(x) = q(a) + q'(a)(x - a) + \frac{q(b) - q(a) - q'(a)(b - a)}{(b - a)^2} (x - a)^2.
\]
Therefore the unique quadratic polynomial that interpolates \( v(a) \), \( v(b) \), and \( v'(a) \) is given by
\[
p(x) = v(a) + v'(a)(x - a) + \frac{v(b) - v(a) - v'(a)(b - a)}{(b - a)^2} (x - a)^2.
\]
Now an application of the Bramble-Hilbert lemma [8] concludes the proof.  **Q.E.D.**

By the lemma, we see that if \( p(a) = v(a) + O(h^2) \), \( p(b) = v(b) + O(h^2) \), and \( p'(a) = v'(a) + O(h) \), then \( |v(x) - p(x)| = O(h^2) \) and \( |v'(x) - p'(x)| = O(h) \) in the entire interval \([a, b]\) including the mid-point.

With the same local angle constraint given in [18] (e.g., at least one angle or complementary angle is larger than or equal to \( \pi/4 \)), we have the following error estimates for the interpolation.

**Proposition 4.1** Assume that \( w = Q = 0 \), and the solution \( u \) is piecewise \( C^2 \), i.e., \( u^\pm \in C(\Omega^\pm) \), then the following error estimates hold:
\[
\|I_h u - u\|_{L^\infty} \leq C h^2 \|D^2 u\|_{L^\infty, \Omega \setminus \Sigma_T}, \quad (4.6)
\]
\[
\|I_h u - u\|_{H^1(\Omega)} \leq C h \|D^2 u\|_{L^\infty, \Omega \setminus \Sigma_T}. \quad (4.7)
\]

For the conforming linear finite element space, the proof is given in [18] already. So we just need to discuss the conforming quadratic finite element space. The proof of the first inequality is straightforward as has been discussed above already. The proof of the second inequality is quite technical and tedious with different geometric configurations. This proof though is quite similar to that for the non-conforming immersed finite element method (cf. [18]), we only give a sketch here. Take the geometry in Fig. 4.4 as an example, we have analytic expression of the interpolation function given in Lemma 4.1 in terms of its values and the tangential derivatives (averaging of that of non-conforming interpolation function). From the analytic expression of the interpolation, it is easy to get \( \|\nabla I_h u - \nabla I_h \tilde{u}\|_{L^\infty} \leq C h \), where \( I_h \tilde{u} \) is any quadratic function when we perturb \( v_A \), \( v_D \), and \( v_E \) by \( O(h^2) \), and \( v'_{AD} \), \( v'_{AE} \), and \( v'_{ED} \) by \( O(h) \).

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Our numerical tests reported in Section 6 confirm such an estimate for the interpolation error. Note that, while the error bounds for the linear and quadratic basis functions are the same, numerical examples of our test problems show that the quadratic one behaves better numerically. Since the quadratic basis function is a conforming one, we can conclude that the finite element solution is first order in $H^1$ norm from the standard finite element theory.

**Remark 1.** Consider a uniform triangular mesh. For both conforming linear and conforming quadratic elements, the basis functions $\phi_i$ associated with the $i$-th node is non-zero only on the six surrounding triangular elements, if the interface does not cut through any of these triangles. Otherwise, the support of a basis function includes two more triangular elements along the direction of the interface, if the interface cuts through any of the surrounding triangle. A corresponding finite-difference scheme, however, has generally a non-standard nine point stencil.

**Remark 2.** If the coefficient $\beta$ is continuous across the interface, i.e., $[\beta]_{\Gamma} = 0$, then both the linear and quadratic basis functions become the standard linear basis functions.

## 5 Implementation

In this section, we give details about the numerical extension of the jumps $w$ and $Q$, the assembly of the stiffness matrix and load vector, and the evaluation of certain integrals on part of an interface element.

### 5.1 Numerical extension of the jumps $w$ and $Q$

We use a Gaussian quadrature to compute the stiffness matrix and the load vector in each triangle like $\triangle ADE$, $\triangle DEB$, and $\triangle BEC$ in Figure 4.2. Let $x$ be such a Gaussian point which is close to the interface. We need to extend the jumps $w$ and $Q$ to this Gaussian point by the definition of $w_{\rho}$ and $Q_{\rho}$ given in $E2$ and $E3$ in Section 2. This is done in two steps. The first step is to find an approximation of the orthogonal projection $\hat{x}$ of $x$ on the interface $\Gamma$, see Lemma 2.1. The second step is to define the extensions of $w$ and $Q$ at $x$ as $w(\hat{x})$ and $Q(\hat{x})$, respectively.

The orthogonal projection can be approximated by

$$\hat{x} = x + \alpha p,$$

where

$$p = \begin{bmatrix} \varphi_x(x) \\ \varphi_y(x) \end{bmatrix}.$$
Here, a subscript denotes a partial derivative. The scalar \( \alpha \) is determined from the following quadratic equation:

\[
\varphi(x) + (\nabla \varphi(x) \cdot p) \alpha + \frac{1}{2} (p^T \text{He}(\varphi(x)) p) \alpha^2 = 0,
\]

where

\[
p^T \text{He}(\varphi) p = \varphi_x^2 \varphi_{xx} + 2 \varphi_x \varphi_y \varphi_{xy} + \varphi_y^2 \varphi_{yy}.
\]

The sign of \( \alpha \) is chosen to be opposite to that of \( \varphi(x) \). If the underlying mesh is uniform, formulated from a Cartesian grid, then, the partial derivatives \( \nabla \varphi(x) \), the Hessian matrix \( \text{He}(\varphi) \) can be computed at \( x \) using the standard centered 5-point finite difference formula. Using the method above, the computed projections have third order of accuracy.

Note that in our implementation, we use the orthogonal projections \( \hat{x} \in \Gamma \). It is also possible to use the orthogonal projections \( \hat{x} \in \Gamma_h \). The difference in the finite-element solution using the two different implementations is small since the area of the mis-matched region \( \sum T_r \) is small. If the level set function is the signed distance function, or a good approximation of the signed distance function, then the error in approximating \( \hat{x} \) would be \( O(h^3) \), see [19]. We also refer the reader to [9,10] for some of the recent discussions on computing orthogonal projections.

Note that when one uses a finite number of piecewise linear basis functions \( \phi_h^i s \in V_h \) to seek the finite element solution, we have introduced an error in substituting \( V = H^1(\Omega) \) by \( V_h \) which is often \( O(h^2) \). Other approximating errors, such as numerical integration, approximating orthogonal projections, have little effect on the finite element solutions as long as they are higher order terms of \( h^3 \).

In practice, we need only to extend both jumps \( w \) and \( Q \) to \( N(\Gamma, 2h) \), the \( 2h \) neighborhood of the interface \( \Gamma \). This is described in (5.4) where the last two terms are identical but differ by a sign for non-interface triangles.

### 5.2 Assembly of stiffness matrix and load vector

On non-interface elements where the interface does not cut through, we can use the standard way in computing the contribution to the stiffness matrix and the load vector. On interface elements where the interface cuts through, we need to modify the load vector (the right-hand side); but the computation for the stiffness matrix remains to be the same as in the case of homogeneous jump conditions.

To fix the idea, let us focus on the quadratic, conforming element (cf. Subsection 4.3). We rewrite the finite-element equation (3.4) in terms of \( u_h = q_h + u_p \):

\[
\int_{\Omega} \beta \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega} f \, v_h \, dx + \int_{\Omega} \beta \nabla u_p \cdot \nabla v_h \, dx + \int_{\Omega} H(\varphi) \nabla \cdot (\beta \nabla u_p) \, v_h \, dx \quad \forall v_h \in V_h.
\]

(5.1)
where $H$ is the Heaviside function and $\tilde{u}_\rho$ is given by

$$
\tilde{u}_\rho(x) = u_\rho(x) + \frac{Q_\rho(x)}{\beta^+(x) |\nabla \varphi(x)|} \quad x \in \Omega.
$$

Note that $\tilde{u}_\rho$ is smooth in a neighborhood of $\Gamma$ and that $u_\rho = \chi_{\Omega^+} \tilde{u}_\rho$ and its corresponding flux have non-homogeneous jumps across the interface.

Let $\{\phi_j(x)\}$ be the basis functions of the modified conforming finite element space. Then, we define

$$
u_h(x) = \left\{ \begin{array}{ll}
\sum_j \alpha_j \phi_j(x) & \text{if } x \text{ belongs to a non-interface triangle}, \\
\sum_j \alpha_j \phi_j(x) + \sum_j u_\rho(x_j) \phi_j(x) & \text{otherwise},
\end{array} \right.
$$

where $\phi_j(x)$ is the $j$-th basis function centered at $x_j$. The left-hand side is exactly the same as in the case of the homogeneous jump condition. The entries of the stiffness matrix are

$$a_{ij} = \int_\Omega \beta \nabla \phi_i \cdot \nabla \phi_j \, dx.$$

Only the right-hand side of the system of equation needs to be modified for certain triangles near the interface.

At a non-interface triangle that is entirely in $\Omega^-$, the last two terms of integration over the triangle are zero, since $H(\varphi(x)) = 0$ and $u_\rho = 0$. The situation is a little more complicated for triangles in $\Omega^+$. If all the non-zero basis functions over a triangle in $\Omega^+$ have no support from interface triangles, then the last two terms in (5.1) are canceled out. To see this, let $T_k$ be such a triangle, and $\phi_t$ be such a basis function with $\Omega_t$ being its support. We have

$$
\int_{\Omega_t} \beta \nabla u_\rho \cdot \nabla \phi_t \, dx + \int_{\Omega_t} H(\varphi) \nabla \cdot (\beta \nabla \tilde{u}_\rho) \phi_t \, dx =
$$

$$= \int_{\Omega_t} \beta \nabla u_\rho \cdot \nabla \phi_t \, dx + \int_{\Omega_t} \nabla \cdot (\beta \nabla \tilde{u}_\rho) \phi_t \, dx
$$

$$= \int_{\Omega_t} \beta \nabla u_\rho \cdot \nabla \phi_t \, dx + \int_{\partial \Omega_t} \frac{\partial \tilde{u}_\rho}{\partial n} \phi_t \, ds - \int_{\Omega_t} \beta \nabla \tilde{u}_\rho \cdot \nabla \phi_t \, dx = 0,
$$

since $\tilde{u}_\rho = u_\rho$ and $\phi_t = 0$ along $\partial \Omega_t$. In other words, the total contribution of the line integral along the boundary of each triangle summed up to be zero. The right hand side
of the load vector can be summarized as the following,

\[
F_i = \begin{cases} 
\int_{\Omega_i} f \phi_i \, dx & \text{if } \Omega^s(\phi_i) \cap \Gamma = \emptyset, \\
\int_{\Omega_i} f \phi_i \, dx + \int_{\Omega} H(\varphi) \nabla \cdot (\beta \nabla \tilde{u}_\rho) \phi_i \, dx 
+ \sum_j u_\rho(x_j) \int_{\Omega_i} \beta \nabla \phi_i \cdot \nabla \phi_j \, dx & \text{otherwise}, 
\end{cases}
\]

(5.4)

where \( \Omega^s(\phi_i) \) is the support of \( \phi_i \).

### 5.3 Evaluation of \( \int_{\Omega} H(\varphi) \nabla \cdot (\beta \nabla \tilde{u}_\rho) \phi_i \, dx \)

Take Figure 4.1 as an example. There are two ways to evaluate the integral. One way is to use

\[
\int_{T_j} H(\varphi) \nabla \cdot (\beta \nabla \tilde{u}_\rho) \phi_i \, dx = \int_{\partial \Delta ADE} \beta \frac{\partial \tilde{u}_\rho}{\partial n} \phi \, ds - \int_{\Delta ADE} \nabla \tilde{u}_\rho \cdot \nabla \phi \, dx 
\]

(5.5)

The line integral is evaluated using a Gaussian quadrature formula or some other numerical quadrature.

The second approach is to evaluate the double integral directly over the triangle using a quadrature formula, say, the four-point formula [14]. The coefficient \( \beta \) is approximated by a constant in the triangle. In order to evaluate the values of the integrand at a given point \( x \), we first find a square \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\) that contains the point \( x \). The Laplacian at the vertices \((x_{i\pm 1}, y_{j\pm 1})\) is computed using the standard three-point central finite difference formula. Finally, the Laplacian at the point \( x \) is interpolated using the bi-linear interpolation, see [17].

### 6 Numerical Results

We report two examples of numerical calculations using our conforming, quadratic, immersed-interface finite-element method. In each of the calculations, we used the ITPACK [23] to solve the resulting linear system of equations.

**Example 1 Homogeneous jump conditions.**

This example is from [18]. We consider the problem (1.1)–(1.5) with \( \Omega = (-1, 1) \times (-1, 1) \), \( \Gamma \) being the circle centered at point \((0,0)\) with radius \( R = 0.5 \), and \( \beta^- = 1 \).
and $\beta^+ = 100$. The source term $f(x, y)$ and the Dirichlet boundary data $u_0(x, y)$ are calculated from the exact solution $u(x, y)$:

$$u(x, y) = \begin{cases} 
\frac{r^3}{\beta^-} & \text{if } r \leq R, \\
\frac{r^3}{\beta^+} + \left( \frac{1}{\beta^-} - \frac{1}{\beta^+} \right) R^3 & \text{otherwise},
\end{cases}$$

where $r = \sqrt{x^2 + y^2}$. The exact solution satisfies the homogeneous jump conditions.

In Table 1(a), we show a grid refinement analysis of our calculations. The first column $E_N = \|u - u_h\|_\infty$ is the error of the finite-element solution measured in the $L^\infty$-norm, and the second column is the estimate of the order of accuracy using the formula

$$\text{order} = \frac{\log (\|E_N\|_\infty / \|E_{2N}\|_\infty)}{\log 2}.$$ 

We see clearly the second order accuracy. The fourth to the last columns are the interpolation errors. We see that the interpolation errors are of second order and its derivatives are of first order. These are optimal.

In Table 1 (b), we show the results and comparison of the finite-element solution in $L^\infty$, $L^2$, and $H^1$ norms. We see second order convergence in $L^\infty$, $L^2$ norms, and first order in $H^1$ norm as expected.

**Example 2** A complicated interface and non-homogeneous jump conditions.

We consider the problem (1.1)-(1.5) with $\Omega = (-1, 1) \times (-1, 1)$ and the interface $\Gamma$ being the zero level set of the function

$$\varphi(x, y) = \sqrt{x^2 + y^2} - 0.1 \sin(5\theta - \pi/5) - 0.5,$$

where $\tan \theta = x/y$ and $0 \leq \theta \leq 2\pi$; see Figure 6.1 for the geometry. The function $\beta$ is defined by

$$\beta(x, y) = \begin{cases} 
x^2 + y^2 + 1 & \text{if } (x, y) \in \Omega^-, \\
b & \text{otherwise,}
\end{cases}$$

where $b > 0$ is a parameter. The function $f$ is defined by

$$f(x, y) = \begin{cases} 
-4 \left(2x^2 + 2y^2 + 1\right) & \text{if } (x, y) \in \Omega^-, \\
2 \sin x \cos y & \text{otherwise.}
\end{cases}$$

Both $\beta$ and $f$ have nonzero jumps across the interface $\Gamma$. The exact solution is

$$u(x, y) = \begin{cases} 
x^2 + y^2, & \text{if } (x, y) \in \Omega^-, \\
\frac{1}{b} \left(\sin x \cos y + \log \sqrt{x^2 + y^2}\right) & \text{otherwise.}
\end{cases}$$
<table>
<thead>
<tr>
<th>$N$</th>
<th>$|u - u_h|_\infty$</th>
<th>$p_1$</th>
<th>$|u - I_h u|_\infty$</th>
<th>$p_2$</th>
<th>$|\frac{\partial u}{\partial x} - \frac{\partial I_h u}{\partial x}|_{\Omega', \infty}$</th>
<th>$p_3$</th>
<th>$|\frac{\partial u}{\partial y} - \frac{\partial I_h u}{\partial y}|_{\Omega', \infty}$</th>
<th>$p_4$</th>
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<td>32</td>
<td>$1.3142 \times 10^{-3}$</td>
<td>2.5895 $10^{-3}$</td>
<td>1.93</td>
<td>1.0559 $10^{-1}$</td>
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<td>5.3514 $10^{-2}$</td>
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<td>2.7544 $10^{-2}$</td>
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<tr>
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</tr>
</tbody>
</table>

Table 1: A grid refinement analysis for Example 1, where $p_i$ are the approximated convergence order and the norms that involve the partial derivatives. (a) The finite-element solution error and the interpolation error. (b) Results and comparison of the finite-element solution errors in $L^\infty(\Omega')$, $L^2(\Omega')$, and $H^1(\Omega')$ norms, where $\Omega' = \Omega \setminus \sum T_r$.

Figure 6.1: The domain and the interface for Example 2.
Notice that both the solution $u$ and the normal flux $\beta \partial_n u$ have nonzero jumps across the interface $\Gamma$.

We remark that the solution behavior depends on the magnitude of the parameter $b$. If $b$ is large, then the solution is close to a piecewise quadratic function. If $b$ is small, then the jumps of the solution and its normal flux across the interface are very large. Numerically, this gives rise to difficulties to achieve optimal convergence properties [1,16]. We test our method for various $b$ and analyze the computed results.

In Table 2, we show a grid refinement analysis for $b = 100$. We see clearly the second-order accuracy in $L^\infty$ and $L^2$ norms, and first-order accuracy in $H^1$ norm.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|u - u_h|<em>\infty / |u|</em>\infty$</th>
<th>$p_1$</th>
<th>$|u - u_h|_{L^2}$</th>
<th>$p_2$</th>
<th>$|u - u_h|_{H^1}$</th>
<th>$p_3$</th>
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<td>1.6705 $\times 10^{-2}$</td>
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<tr>
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<td>$9.466 \times 10^{-6}$</td>
<td>2.44</td>
<td>$2.531 \times 10^{-2}$</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Table 2: A grid refinement analysis for Example 2 with $b = 100$, where $p_i$ are the approximated convergence order and the norms that involve the partial derivatives. Second-order accuracy in $L^\infty$ norm is observed.

Where $b$ gets smaller, the jumps in the solution and flux get larger. For interface problems, the errors obtained from non-body fitted meshes usually do not decrease monotonically as we refine the mesh; see for example [15]. For small $b$, it is thus more realistic to find the asymptotic convergence rate as the slope of the line fitting of the experimental data ($\log(h_i), \log(E_i)$).

In Figure 6.2, we show the linear regression analysis for $b = 1$ and $b = 0.1$ for the computed finite element solution. For these two cases, the convergence orders are 2.8122; and 2.4061. As the mesh gets finer, the linear regression analysis done (by deleting the results from coarse meshes) are getting closer to number two indicating a second-order accuracy. In Figure 6.3, we also show the linear regression analysis in $L^2$ and $H^1$ norm. The convergence orders are 1.9906; and 0.9135 respectively.

Figure 6.4 shows the result for $b = 0.01$ that is quite small, and hence there is large ratio in the coefficient from both sides of the interface. The convergence order from the sample meshes ranging from 40 to 500 with 10 increment is 1.8875, see Figure 6.4 (a). But as the mesh gets finer, the linear regression analysis done by cutting the results from coarse meshes are getting closer to number two again indicating a second-order accuracy. Figure 6.4 (b) shows the convergence order to be 1.9811.
Figure 6.2: The linear regression analysis in the $L^\infty$ norm in log-log scale with the mesh varying according to $N = 40 + 20k$, $k = 0, 1, \ldots, 23$. (a) $b = 1$, the slope (convergence order) is 2.8122; (b) $b = 0.1$, the slope is 2.4061.

Figure 6.3: The linear regression analysis in the $L^2$ norm (a), and in $H^1$ norm (b), in log-log scale with the mesh varying according to $N = 40 + 20k$, $k = 0, 1, \ldots, 23$, $b = 1$. The slope (convergence order) is 1.9906 and 0.9135, respectively.
Figure 6.4: The linear regression analysis in the $L^\infty$ norm in log-log scale for $b = 0.01$. (a). $N = 40 + 20k$, $k = 0, 1, \cdots, 23$, the slope (convergence order) is 1.8875; (b). $N = 40 + 20k$, $k = 1, 15, \cdots, 23$, the slope is 1.9811.

7 Conclusions

We have developed a class of immersed-interface finite-element methods for solving elliptic interface problems with non-homogeneous jump conditions. These methods consist of three parts:

(a) A weak formulation of the problem in which the non-homogeneous jump conditions are removed by using the level-set representation of the interface;

(b) Construction of immerse-interface finite-element basis functions for irregular nodes that satisfy the homogeneous jump conditions;

(c) Several techniques of numerical implementation for the resulting finite-element equations.

Our methods have several advantages. For instance, they result symmetry positive definite systems of linear equations. Moreover, they can be used with the level-set method for fast simulations of interface dynamics. Our basic error analysis and numerical tests demonstrate that such methods have optimal convergence properties.

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References


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