

VARIATIONAL PROPERTIES OF UNBOUNDED ORDER PARAMETERS*

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Abstract. Order parameters in physical and biological systems can sometimes become unbounded as the size of an underlying system increases. It is proposed that such a quantity be modeled as a minimizer of the energy functional

$$I_\varepsilon(u) = \int \left[\frac{\varepsilon^2}{2} |\nabla u|^2 - \frac{1}{2} \log(1 + |u|^2) \right] dx,$$

where u is constrained by a side condition, and $\varepsilon > 0$ is a parameter that is inversely proportional to the linear size of the system. It is shown that a minimizer of I_ε exists; the minimum value of I_ε scales as $\log \varepsilon$; and both the L^2 and H^1 norms of any minimizer of I_ε are of the order $O(1/\varepsilon)$, indicating the unboundedness of the order parameter. It is also shown that the renormalized energy functionals

$$J_\varepsilon(v) = I_\varepsilon\left(\frac{v}{\varepsilon}\right) - \log \varepsilon$$

Γ -converge to the functional

$$J(v) = \int \left(\frac{1}{2} |\nabla v|^2 - \log |v| \right) dx.$$

Minimizers of this Γ -limit for scalar order parameters with the Dirichlet boundary condition are well characterized.

Key words. order parameters, variational models, energy asymptotics, renormalized energy, Γ -convergence

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1. Introduction. Order parameters in physical and biological systems, such as population, concentration, volume fractions, magnetization vectors, directors of liquid crystals, the slope of surface height profile of thin films, etc., are mathematically scalar or vector-valued functions, or gradients of functions. An order parameter can sometimes grow unbounded as the size of an underlying system increases. An example of such an unbounded order parameter is the slope of surface of an epitaxially growing thin film in some experimental situations [5, 12, 20].

We propose to model unbounded order parameters as possible minimizers or low energy configurations of the effective free energy functional

$$(1.1) \quad \hat{I}(\hat{u}) = \int_{\hat{\Omega}} \left[\frac{\alpha}{2} |\nabla \hat{u}|^2 - \frac{\beta}{2} \log(1 + |\hat{u}|^2) \right] d\hat{x},$$

where $\hat{\Omega} \subset \mathbb{R}^n$ for some integer $n \geq 1$ is a bounded domain, $\alpha > 0$ and $\beta > 0$ are two material constants, and the functions $\hat{u} : \hat{\Omega} \rightarrow \mathbb{R}^m$ with some integer $m \geq 1$ are constrained by a boundary condition or some other side conditions. Here and

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below, we denote $f_E = \frac{1}{|E|} \int_E$ for a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ with a finite and nonzero Lebesgue measure $|E|$. For any $a = (a_1, \dots, a_m) \in \mathbb{R}^m$, we denote $|a| = \sqrt{\sum_{i=1}^m a_i^2}$. For a differentiable function $u = (u_1, \dots, u_m) : D \rightarrow \mathbb{R}^m$ with $D \subset \mathbb{R}^n$ an open set, we define $\nabla u : D \rightarrow \mathbb{R}^{m \times n}$ to be the $m \times n$ -matrix-valued function with $\partial_{x_j} u_i(x)$ the (i, j) -entry of $\nabla u(x)$ and denote $|\nabla u(x)| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |\partial_{x_j} u_i(x)|^2}$ for $x = (x_1, \dots, x_n) \in D$.

In the special case with $\hat{u} = \nabla \hat{h}$ for some scalar function \hat{h} defined on a two-dimensional domain, the functional (1.1) happens to be the Liapunov functional of the evolution equation

$$(1.2) \quad \frac{\partial \hat{h}}{\partial t} = -\alpha \Delta^2 \hat{h} - \beta \nabla \cdot \left(\frac{\nabla \hat{h}}{1 + |\nabla \hat{h}|^2} \right),$$

i.e., this equation is the gradient-flow induced by the functional (1.1). Equation (1.2) was first proposed phenomenologically in [12] to model the surface height profile \hat{h} , measured in a comoving frame, in epitaxial growth of thin films with a strong asymmetry of the adatom (adsorbed atom) attachment and detachment from lower and upper terraces to atomic step edges due to the existence of an energy barrier [4, 17, 18]. Numerical and analytical studies based on such a model have shown that the slope of the surface, $|\nabla \hat{h}|$, which is the order parameter in this case, grows unbounded, agreeing with experiments [10, 11, 12, 13, 15, 16, 21].

It is interesting to compare the energy functional (1.1) with a usual Ginzburg–Landau-type energy functional that has the term $(|\hat{u}|^2 - 1)^2$ or alike, instead of the negative logarithmic term in (1.1). Important examples of the latter include the Ginzburg–Landau energy for superconductors [3, 9] and the Cahn–Hilliard energy for phase separation [2]; both have been much studied. It is obvious that an order parameter modeled by a Ginzburg–Landau-type energy functional stays always bounded. If $|\hat{u}| \ll 1$, then by the Taylor expansion,

$$-\log(1 + |\hat{u}|^2) = -|\hat{u}|^2 + \frac{1}{2}|\hat{u}|^4 + O(|\hat{u}|^6) = \frac{1}{2}(|\hat{u}|^2 - 1)^2 - \frac{1}{2} + O(|\hat{u}|^6).$$

Thus, both types of energy functionals have approximately the same energy landscape for admissible functions with very small magnitude. As a consequence, the zero function as a critical point is unstable in both types of models.

Intuitively, if the energy $\hat{I}(\hat{u})$ of an admissible function $\hat{u} : \hat{\Omega} \rightarrow \mathbb{R}^m$ is very small, then the magnitude $|\hat{u}|$ of the function must be very large in some norm. But, the boundary condition (or other side conditions) and the presence of the gradient term in the energy $\hat{I}(\hat{u})$ prevent $|\hat{u}|$ from being too large. These competing mechanisms determine the magnitude of such a low energy function to be finite but to grow unbounded as the system size increases. Our primary goals of this work are to quantify such unboundedness and to characterize the asymptotic behavior of energy functionals for systems of large size.

We shall not, however, directly work with the functional \hat{I} defined in (1.1). Rather, we shall first rescale the energy functional. The idea is clear for the special case that $\hat{\Omega} = (0, \hat{L})^n$, a cube in \mathbb{R}^n of linear size $\hat{L} > 0$: letting $u(x) = \hat{u}(\hat{x})$ with $x = \hat{L}^{-1}\hat{x}$, one obtains that $I_\varepsilon(u) = \beta \hat{I}(\hat{u})$, where

$$(1.3) \quad I_\varepsilon(u) = \int_{\Omega} \left[\frac{\varepsilon^2}{2} |\nabla u|^2 - \frac{1}{2} \log(1 + |u|^2) \right] dx,$$

$\varepsilon = \sqrt{\alpha/\beta}\widehat{L}^{-1}$, and Ω is the unit cube of \mathbb{R}^n . Now, for a general bounded domain $\widehat{\Omega}$, one can fix some point $\hat{x}_0 \in \widehat{\Omega}$ and apply the change of variable $\hat{x} \rightarrow x = \widehat{L}^{-1}(\hat{x} - \hat{x}_0)$ with \widehat{L} being the diameter of $\widehat{\Omega}$. One again obtains an equivalent variational problem with the energy functional given by (1.3), in which ε is inversely proportional to \widehat{L} and $\Omega \subset \mathbb{R}^n$ is a fixed bounded domain whose diameter is independent of \widehat{L} .

Depending on how an underlying physical and biological system is modeled mathematically, the set of admissible functions, to be denoted by $\mathcal{H}(\Omega, \mathbb{R}^m)$, for the energy functional I_ε can be defined differently. In this work, we assume that $\Omega \subset \mathbb{R}^n$ in the definition of I_ε is a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, and define

$$(1.4) \quad \mathcal{H}(\Omega, \mathbb{R}^m) = H_0^1(\Omega, \mathbb{R}^m),$$

or

$$(1.5) \quad \mathcal{H}(\Omega, \mathbb{R}^m) = \left\{ u \in H^1(\Omega, \mathbb{R}^m) : \int_{\Omega} u \, dx = 0 \right\},$$

where $H^1(\Omega, \mathbb{R}^m)$ and $H_0^1(\Omega, \mathbb{R}^m)$ are the spaces of vector-valued functions whose components are in the usual Sobolev spaces of scalar functions $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively [1, 8]. In both cases, $\mathcal{H}(\Omega, \mathbb{R}^m)$ is a closed subspace of the Hilbert space $H^1(\Omega, \mathbb{R}^m)$ that is equipped with the norm $\|u\| = \sqrt{\|u\|^2 + \|\nabla u\|^2}$ for all $u \in H^1(\Omega, \mathbb{R}^m)$, where $\|\cdot\|$ denotes the $L^2(\Omega)$ -norm.

Our major results are as follows:

(1) For each $\varepsilon > 0$, there exists a minimizer of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. Moreover, the minimum energy scales as $\log \varepsilon$, and both the L^2 and H^1 norms of any minimizer of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ are of the order $O(1/\varepsilon)$, cf. Theorem 2.1;

(2) The renormalized energy functionals $J_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$, defined by

$$(1.6) \quad J_\varepsilon(v) = I_\varepsilon\left(\frac{v}{\varepsilon}\right) - \log \varepsilon \quad \forall v \in \mathcal{H}(\Omega, \mathbb{R}^m),$$

Γ -converge to the energy functional $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$(1.7) \quad J(v) = \int_{\Omega} \left(\frac{1}{2} |\nabla v| - \log |v| \right) dx, \quad v \in \mathcal{H}(\Omega, \mathbb{R}^m),$$

cf. Theorem 3.2. Moreover, if $v_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$ for $\varepsilon > 0$ is a minimizer of $J_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$, then there exists a subsequence of $\{v_\varepsilon\}_{\varepsilon > 0}$ that converges *strongly* in $\mathcal{H}(\Omega, \mathbb{R}^m)$ to a minimizer of $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$, cf. Theorem 3.3;

(3) In the case of the scalar Dirichlet boundary-value problem, there exists a unique $v_+ \in H_0^1(\Omega)$ such that v_+ is smooth and positive in Ω , and that v_+ and $-v_+$ are the only minimizers of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, cf. Theorem 4.1.

The results in (1) and (2) hold also true if the set of admissible functions is $\{u \in H_0^1(\Omega, \mathbb{R}^m) : \int_{\Omega} u \, dx = 0\}$, or if Ω is an open cube in \mathbb{R}^n with its faces parallel to the coordinate planes and the corresponding set of admissible functions is $\{u \in H_{\text{per}}^1(\Omega, \mathbb{R}^m) : \int_{\Omega} u \, dx = 0\}$, where $H_{\text{per}}^1(\Omega, \mathbb{R}^m)$ is the closure in $H^1(\Omega, \mathbb{R}^m)$ of the set of all C^∞ , $\overline{\Omega}$ -periodical functions from \mathbb{R}^n to \mathbb{R}^m .

The heuristics behind the first part of our results is well illustrated in our previous work [13] through the calculation of trial functions with low energy using an ad hoc ansatz and the calculation of critical points of the energy functional using matched

asymptotics, with both calculations being done in a one-dimensional setting. Our results in (1) generalize those in [13] for more complicated domains and include the optimal lower bound as well as the precise asymptotics of the minimum energy.

Our results do not directly apply to continuum models, such as the Liapunov functional of the equation (1.2), of the epitaxial growth with a significant attachment-detachment asymmetry of adatoms. This is because the set of admissible functions \hat{u} of the functional (1.1) is larger than the set of gradient vector fields. However, the approach developed in this work can be applied to the study of such continuum models and to obtain similar results. In particular, the large-system-size Γ -limit of the rescaled Liapunov functionals—the functional (1.3) with u replaced by ∇h for the surface height function h that models a finite energy barrier—is precisely Villain’s model for an infinite energy barrier [21].

Potentially, the positive solution to the scalar Dirichlet boundary-value problem for the limiting functional can be a good alternative to the distance function in the reinitialization process of the widely used level-set numerical method [14, 19]. We will address these issues of application in separate works.

In section 2, we present and prove the results for the energy functionals $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. In section 3, we prove the Γ -convergence of the renormalized energies $J_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ to the functional $J : \mathcal{H}(\Omega, \mathbb{R}) \rightarrow \mathbb{R} \cup \{\infty\}$. Finally, in section 4, we characterize solutions to the scalar Dirichlet problem of infimizing the limiting energy defined in (1.7).

2. Energy asymptotics and bounds of energy minimizers. We consider the energy functionals $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$, defined in (1.3) for a general domain Ω , only for $\varepsilon \in (0, 1]$, though many of our results hold true also for any $\varepsilon > 0$. For convenience, we denote $\|u\| = \sqrt{\int_\Omega |u(x)|^2 dx} = (1/\sqrt{|\Omega|})\|u\|$ for all $u \in L^2(\Omega)$.

The following is our main result in this section:

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. Let $\mathcal{H}(\Omega, \mathbb{R}^m)$ be defined as in (1.4) or (1.5).*

(1) *For each $\varepsilon \in (0, 1]$, there exists $u_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$ such that*

$$(2.1) \quad I_\varepsilon(u_\varepsilon) = \min_{u \in \mathcal{H}(\Omega, \mathbb{R}^m)} I_\varepsilon(u).$$

(2) *Let $\mu_\varepsilon = \min_{u \in \mathcal{H}(\Omega, \mathbb{R}^m)} I_\varepsilon(u)$. There exist constants C_1 and C_2 that depend only on Ω such that*

$$(2.2) \quad C_1 + \log \varepsilon \leq \mu_\varepsilon \leq C_2 + \log \varepsilon \quad \forall \varepsilon \in (0, 1].$$

Moreover, $\mu_\varepsilon - \log \varepsilon$ increases as $\varepsilon \in (0, 1]$ decreases, $\nu := \sup_{0 < \varepsilon \leq 1} (\mu_\varepsilon - \log \varepsilon)$ is finite, and

$$(2.3) \quad \lim_{\varepsilon \rightarrow 0^+} (\mu_\varepsilon - \log \varepsilon) = \nu.$$

(3) *There exist constants $C_j > 0$ ($j = 3, 4, 5, 6$) and $\varepsilon_0 \in (0, 1]$, all depending only on Ω , such that for any minimizer $u_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$ of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ and for all $\varepsilon \in (0, \varepsilon_0]$,*

$$(2.4) \quad \frac{C_3}{\varepsilon} \leq \|u_\varepsilon\| \leq \frac{C_4}{\varepsilon},$$

$$(2.5) \quad \frac{C_5}{\varepsilon} \leq \|\nabla u_\varepsilon\| \leq \frac{C_6}{\varepsilon}.$$

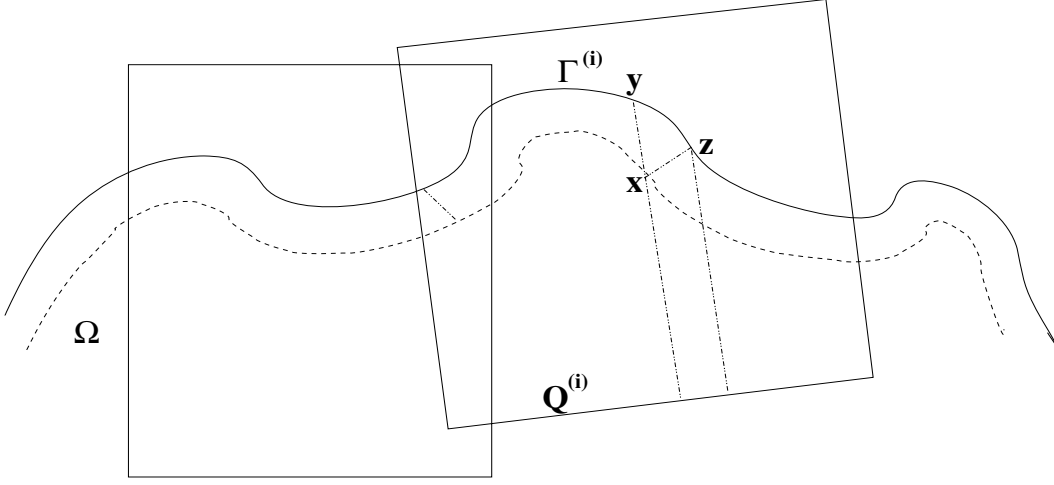


FIG. 2.2.1.

To prove this theorem, we need some preparations. We recall for any compact set $S \subset \mathbb{R}^n$ that the distance function $\text{dist}(\cdot, S) : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$\text{dist}(x, S) = \inf_{y \in S} |x - y| = \min_{y \in S} |x - y| \quad \forall x \in \mathbb{R}^n,$$

is a Lipschitz-continuous function:

$$(2.6) \quad |\text{dist}(x, S) - \text{dist}(y, S)| \leq |x - y| \quad \forall x, y \in \mathbb{R}^n.$$

Moreover, it is differentiable almost everywhere in \mathbb{R}^n , and

$$(2.7) \quad |\nabla \text{dist}(x, S)| = 1 \quad \text{a.e. } x \in \mathbb{R}^n,$$

cf. the proof of Lemma 3.2.34 in [7].

LEMMA 2.1. *If $\Omega \in \mathbb{R}^n$ is a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, then there exist constants $s_0 > 0$ and $C_0 > 0$, both depending only on Ω , such that*

$$|\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq s\}| \leq C_0 s \quad \forall s \in (0, s_0].$$

Proof. Since $\partial\Omega$ is Lipschitz-continuous, there exist finitely many Lipschitz-continuous functions $\phi^{(i)} : Q^{(i)} \rightarrow \mathbb{R}$ ($i = 1, \dots, m$ for some integer $m \geq 1$) in local Cartesian coordinates with each $Q^{(i)} = \prod_{j=1}^{n-1} [-\alpha_j^{(i)}, \alpha_j^{(i)}]$ for some $\alpha_j^{(i)} > 0$ ($1 \leq i \leq m$ and $1 \leq j \leq n-1$) a cube in \mathbb{R}^{n-1} , that satisfy the following properties (cf. Figure 2.2.1):

(1) For each i with $1 \leq i \leq m$, the local Cartesian coordinates $\xi^{(i)} = (\xi_1^{(i)}, \dots, \xi_n^{(i)})$ are obtained by rotating and translating the original Cartesian coordinates $x = (x_1, \dots, x_n)$;

(2) There exist $\alpha_n > 0$ and $\beta_n > 0$ such that for each cube $G^{(i)} := Q^{(i)} \times [-\alpha_n, \alpha_n] \subset \mathbb{R}^n$ ($1 \leq i \leq m$),

$$\Gamma^{(i)} := G^{(i)} \cap \partial\Omega = \left\{ \left(\hat{\xi}^{(i)}, \xi_n^{(i)} \right) \in \mathbb{R}^n : \hat{\xi}^{(i)} \in Q^{(i)}, \xi_n^{(i)} = \phi^{(i)} \left(\hat{\xi}^{(i)} \right) \right\},$$

where $\hat{\xi}^{(i)} = (\xi_1^{(i)}, \dots, \xi_{n-1}^{(i)})$, and

$$U_+^{(i)} := \left\{ (\hat{\xi}^{(i)}, \xi_n^{(i)}) \in \mathbb{R}^n : \hat{\xi}^{(i)} \in Q^{(i)}, \phi^{(i)}(\hat{\xi}^{(i)}) < \xi_n^{(i)} < \phi^{(i)}(\hat{\xi}^{(i)}) + \beta_n \right\} \subset G^{(i)} \cap \bar{\Omega}^c,$$

$$U_-^{(i)} := \left\{ (\hat{\xi}^{(i)}, \xi_n^{(i)}) \in \mathbb{R}^n : \hat{\xi}^{(i)} \in Q^{(i)}, \phi^{(i)}(\hat{\xi}^{(i)}) - \beta_n < \xi_n^{(i)} < \phi^{(i)}(\hat{\xi}^{(i)}) \right\} \subset G^{(i)} \cap \Omega;$$

(3) The union $\cup_{i=1}^m G^{(i)}$ covers a neighborhood of the compact set $\partial\Omega$, and $\partial\Omega = \cup_{i=1}^m \Gamma_i$.

Now, since the distance function $\text{dist}(\cdot, \partial\Omega) : \bar{\Omega} \rightarrow \mathbb{R}$ is Lipschitz-continuous and vanishes only on the boundary $\partial\Omega$ which is compact, by properties (2) and (3), there exists a constant $s_1 = s_1(\Omega) > 0$ such that

$$\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq s_1\} \subseteq \bigcup_{i=1}^m U_-^{(i)}.$$

By this and property (3), there exist cubes $P^{(i)}$ in \mathbb{R}^{n-1} with $P^{(i)} \subseteq Q^{(i)}$ ($1 \leq i \leq m$) and a constant $s_0 = s_0(\Omega)$ with $0 < s_0 \leq s_1$ that satisfy the following properties:

(4) For each i ($1 \leq i \leq m$),

$$\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq s_0\} \subseteq \bigcup_{i=1}^m V^{(i)},$$

where

$$V^{(i)} = \left\{ (\hat{\xi}^{(i)}, \xi_n^{(i)}) \in \mathbb{R}^n : \hat{\xi}^{(i)} \in \hat{P}^{(i)}, \phi^{(i)}(\hat{\xi}^{(i)}) - \beta_n < \xi_n^{(i)} < \phi^{(i)}(\hat{\xi}^{(i)}) \right\} \subseteq U_-^{(i)};$$

(5) Let $x \in \Omega$ and $x' \in \partial\Omega$ be such that $\text{dist}(x, \partial\Omega) = |x - x'| \leq s_0$. If $x \in V^{(i)}$ for some i ($1 \leq i \leq m$), then $x' \in \Gamma_i = G^{(i)} \cap \partial\Omega$.

Fix $s \in \mathbb{R}$ with $0 < s \leq s_0$. Let $x \in \Omega$ be such that $\text{dist}(x, \partial\Omega) \leq s$. By property (4), we have $x \in V^{(i)}$ for some i with $1 \leq i \leq m$, cf. Figure 2.1. Let $\xi^{(i)} = (\hat{\xi}^{(i)}, \xi_n^{(i)})$ be the local coordinates of x in which $\hat{\xi}^{(i)} \in P^{(i)} \subseteq Q^{(i)}$. Let the point $y \in \mathbb{R}^n$ have the local coordinates $(\hat{\xi}^{(i)}, \phi^{(i)}(\hat{\xi}^{(i)}))$. Then, by property (2), $y \in \Gamma^{(i)} \subset \partial\Omega$, cf. Figure 2.1. Let $z \in \partial\Omega$ be such that $\text{dist}(x, \partial\Omega) = |x - z| \leq s$, cf. Figure 2.1. By property (5), $z \in \Gamma^{(i)} = G^{(i)} \cap \partial\Omega$. Thus, there exists $(\hat{\eta}^{(i)}, \eta_n^{(i)}) \in \Gamma^{(i)} = G^{(i)} \cap \partial\Omega$ such that $\hat{\eta}^{(i)} \in Q^{(i)}$, $(\hat{\eta}^{(i)}, \eta_n^{(i)}) = (\hat{\eta}^{(i)}, \phi^{(i)}(\hat{\eta}^{(i)}))$ are the local coordinates of z , and

$$(2.8) \quad \text{dist}(x, \partial\Omega) = |x - z| = \left| (\hat{\xi}^{(i)}, \xi_n^{(i)}) - (\hat{\eta}^{(i)}, \phi^{(i)}(\hat{\eta}^{(i)})) \right| \leq s.$$

This implies that

$$(2.9) \quad \left| \hat{\xi}^{(i)} - \hat{\eta}^{(i)} \right| \leq s.$$

Denoting by $L_i > 0$ the Lipschitz constant of the Lipschitz-continuous function $\phi^{(i)} : Q^{(i)} \rightarrow \mathbb{R}$, we have by (2.8) and (2.9) that

$$\begin{aligned} \left| \phi^{(i)}(\hat{\xi}^{(i)}) - \xi_n^{(i)} \right| &= |x - y| \\ &\leq |x - z| + |y - z| \\ &\leq s + \left| \hat{\xi}^{(i)} - \hat{\eta}^{(i)} \right| + \left| \phi^{(i)}(\hat{\xi}^{(i)}) - \phi^{(i)}(\hat{\eta}^{(i)}) \right| \\ &\leq s + s + L_i \left| \hat{\xi}^{(i)} - \hat{\eta}^{(i)} \right| \leq (2 + L_i)s. \end{aligned}$$

The arbitrariness of x now implies that

$$\begin{aligned} & \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq s\} \\ & \subseteq \bigcup_{i=1}^m \left\{ \left(\hat{\xi}^{(i)}, \xi_n^{(i)} \right) : \hat{\xi}^{(i)} \in \hat{Q}^{(i)}, \phi^{(i)}(\hat{\xi}^{(i)}) - (2 + L_i)s \leq \xi_n^{(i)} \leq \phi^{(i)}(\hat{\xi}^{(i)}) \right\}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} & |\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq s\}| \\ & \leq \sum_{i=1}^m \int_{\hat{Q}^{(i)}} \left[\phi^{(i)}(\hat{\xi}^{(i)}) - \left(\phi^{(i)}(\hat{\xi}^{(i)}) - (2 + L_i)s \right) \right] d\hat{\xi}^{(i)} \\ & \leq C_0 s, \end{aligned}$$

where $C_0 = \sum_{i=1}^m (2 + L_i) |Q^{(i)}| > 0$, depending only on Ω , and $|Q^{(i)}|$ is the $(n-1)$ -dimensional volume of $Q^{(i)}$ ($1 \leq i \leq m$). \square

LEMMA 2.2. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ that has a Lipschitz-continuous boundary $\partial\Omega$, there exists a Lipschitz-continuous function $f : \bar{\Omega} \rightarrow \mathbb{R}$ such that*

$$f = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} f \, dx = 0, \quad \text{and} \quad -\infty < \int_{\Omega} \log |f| \, dx < \infty.$$

Proof. Let $x_0 \in \Omega$ and $\rho > 0$ be such that the ball $B := B(x_0, \rho) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ is completely contained in Ω , i.e., $\bar{B} \subset \Omega$. For any $x \in \bar{\Omega}$, let $d(x)$ be the distance from x to the compact set $\partial\Omega \cup \partial B$ and define $f : \bar{\Omega} \rightarrow \mathbb{R}$ by

$$(2.10) \quad f(x) = \begin{cases} d(x) & \text{if } x \in \bar{\Omega} \setminus B, \\ -\gamma d(x) & \text{if } x \in B, \end{cases}$$

where $\gamma = \int_{\Omega \setminus B} d(x) \, dx / \int_B d(x) \, dx > 0$. Clearly, $f : \bar{\Omega} \rightarrow \mathbb{R}$ is continuous, $f = 0$ on $\partial\Omega$, and $\int_{\Omega} f(x) \, dx = 0$.

We show now that $f : \bar{\Omega} \rightarrow \mathbb{R}$ is Lipschitz-continuous. Fix $x, y \in \bar{\Omega}$. If both x and y are in B or both x and y are in $\bar{\Omega} \setminus B$, then we have by (2.6) and (2.10) that

$$(2.11) \quad |f(x) - f(y)| \leq \max(1, \gamma) |x - y| \leq (1 + \gamma) |x - y|.$$

Assume now $x \in B$ but $y \in \bar{\Omega} \setminus B$. Choose $\delta \in \mathbb{R}$ so that $0 < \delta < 1$, the ball $B_1 := B(x_0, \rho + \delta) \subset \Omega$, and

$$(2.12) \quad 0 < 2\delta < \text{dist}(\partial\Omega, \bar{B}) := \inf_{x' \in \partial\Omega, y' \in \bar{B}} |x' - y'|.$$

If $y \in \bar{\Omega} \setminus B_1$, then $|x - y| \geq \delta$. Hence,

$$(2.13) \quad |f(x) - f(y)| \leq \frac{2 \max_{z \in \bar{\Omega}} |f(z)|}{\delta} |x - y|.$$

If $y \in B_1 \setminus B$, then

$$d(y) = \text{dist}(y, \partial B) \leq |x - y|.$$

Also,

$$d(x) = \text{dist}(x, \partial B) \leq |x - y|.$$

Thus,

$$(2.14) \quad |f(x) - f(y)| = \gamma d(x) + d(y) \leq (1 + \gamma)|x - y|.$$

Setting

$$L = \max \left(1 + \gamma, \frac{2 \max_{z \in \bar{\Omega}} |f(z)|}{\delta} \right) > 0,$$

we obtain from (2.11), (2.13), and (2.14) that

$$|f(x) - f(y)| \leq L|x - y|.$$

Since $x, y \in \bar{\Omega}$ are arbitrary, the function $f : \bar{\Omega} \rightarrow \mathbb{R}$ is Lipschitz-continuous.

We show finally that $-\infty < \int_{\Omega} \log |f| dx < \infty$. Since $|f|$ is bounded from above on $\bar{\Omega}$,

$$\begin{aligned} \int_{\Omega} \log |f| dx &= \int_{\{x \in \Omega : |f(x)| < 1\}} \log |f| dx + \int_{\{x \in \Omega : |f(x)| \geq 1\}} \log |f| dx \\ &\leq \int_{\{x \in \Omega : |f(x)| \geq 1\}} \log |f| dx < \infty. \end{aligned}$$

So, we need only to show that

$$(2.15) \quad \int_{\Omega} \log |f| dx = \int_{B_1} \log |f| dx + \int_{\Omega \setminus B_1} \log |f| dx > -\infty,$$

where $B_1 = B(x_0, \rho + \delta) \subset \Omega$ is the same ball used before and $\delta > 0$ is given in (2.12).

Using (2.10) and (2.12), the polar coordinates, and a change of variables, we obtain

$$\begin{aligned} \int_{B_1} \log |f| dx &= \int_B \log(\gamma d(x)) dx + \int_{B_1 \setminus B} \log d(x) dx \\ &= |B| \log \gamma + \int_B \log |\rho - |x - x_0|| dx + \int_{B_1 \setminus B} \log |\rho - |x - x_0|| dx \\ &= |B| \log \gamma + S_n \int_0^{\rho} r^{n-1} \log |\rho - r| dr + S_n \int_{\rho}^{\rho+\delta} r^{n-1} \log |\rho - r| dr \\ (2.16) \quad &> -\infty, \end{aligned}$$

where S_n is the surface area of the unit ball in \mathbb{R}^n .

Observe that for $x \in \Omega \setminus B_1$ with $d(x) < \delta$, we have by (2.12) that in fact $d(x) = \text{dist}(x, \partial\Omega)$. Thus, by Lemma 2.1, there exists an integer $N \geq 1$ and a constant $C_0 > 0$ such that $|\omega_j| \leq C_0 \delta 2^{-j}$ ($j = N, \dots$), where

$$\begin{aligned} \omega_j &:= \{x \in \Omega \setminus B_1 : 2^{-(j+1)}\delta < d(x) \leq 2^{-j}\delta\} \\ &\subset \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq 2^{-j}\delta\}, \quad j = 0, \dots \end{aligned}$$

Setting $E_{\delta} = \{x \in \Omega \setminus B_1 : d(x) > \delta\}$, we see that $\Omega \setminus B_1$ is the union of the pairwise

disjoint sets E_δ and ω_j ($j = 0, \dots$). Therefore, by the fact that $0 < \delta < 1$, we obtain

$$\begin{aligned}
\int_{\Omega \setminus B_1} \log |f| dx &= \int_{E_\delta} \log d(x) dx + \sum_{j=0}^{\infty} \int_{\omega_j} \log d(x) dx \\
&\geq |\Omega \setminus B_1| \log \delta + \sum_{j=0}^N |\omega_j| \log \left(2^{-(j+1)} \delta \right) + \sum_{j=N+1}^{\infty} |\omega_j| \log \left(2^{-(j+1)} \delta \right) \\
&\geq |\Omega \setminus B_1| \log \delta + |\Omega \setminus B_1| \log \left(2^{-(N+1)} \delta \right) \\
&\quad + C_0 \delta \sum_{j=N+1}^{\infty} 2^{-j} \log \left(2^{-(j+1)} \delta \right) \\
&= |\Omega \setminus B_1| \log \left(2^{-(N+1)} \delta^2 \right) + C_0 \delta \sum_{j=N+1}^{\infty} 2^{-j} [\log \delta - (j+1) \log 2] \\
(2.17) \quad &> -\infty.
\end{aligned}$$

Finally, (2.15) follows from (2.16) and (2.17). \square

We are now ready to prove our main result in this section.

Proof of Theorem 2.1.

(1) Fix $\varepsilon \in (0, 1]$. Recall the Poincaré inequality [1, 6, 8]

$$(2.18) \quad \|u\| \leq C_0 \|\nabla u\| \quad \forall u \in \mathcal{H}(\Omega, \mathbb{R}^m),$$

where $C_0 > 0$ is a constant depending only on Ω . Since $(1/s) \log(1+s) \rightarrow 0$ as $s \rightarrow \infty$, there exists $R_\varepsilon = R_\varepsilon(\Omega) > 0$ such that

$$(2.19) \quad \log(1+s) \leq \frac{\varepsilon^2 s}{2C_0^2} \quad \forall s \geq R_\varepsilon.$$

By (2.19) and (2.18), we have

$$\begin{aligned}
I_\varepsilon(u) &= \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: |u|^2 \leq R_\varepsilon\}} \log(1+|u|^2) dx \\
&\quad - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: |u|^2 > R_\varepsilon\}} \log(1+|u|^2) dx \\
&\geq \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: |u|^2 \leq R_\varepsilon\}} \log(1+R_\varepsilon) dx \\
&\quad - \frac{\varepsilon^2}{4C_0^2 |\Omega|} \int_{\{x \in \Omega: |u|^2 > R_\varepsilon\}} |u|^2 dx \\
&\geq \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \log(1+R_\varepsilon) - \frac{\varepsilon^2}{4C_0^2} \int_{\Omega} |u|^2 dx \\
(2.20) \quad &\geq \frac{\varepsilon^2}{4} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \log(1+R_\varepsilon) \quad \forall u \in \mathcal{H}(\Omega, \mathbb{R}^m).
\end{aligned}$$

Set $\mu_\varepsilon = \inf_{u \in \mathcal{H}(\Omega, \mathbb{R}^m)} I_\varepsilon(u)$. By (2.20), $\mu_\varepsilon > -\infty$. Let $\{u_j\}_{j=1}^{\infty}$ be an infimizing sequence of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. It follows from (2.20) and (2.18) that $\{u_j\}_{j=1}^{\infty}$ is bounded in $\mathcal{H}(\Omega, \mathbb{R}^m)$. Thus, up to a subsequence, $u_j \rightharpoonup u_\varepsilon$ in $H^1(\Omega, \mathbb{R}^m)$ and

$u_j \rightarrow u_\varepsilon$ in $L^2(\Omega, \mathbb{R}^m)$ as $j \rightarrow \infty$ for some $u_\varepsilon \in H^1(\Omega, \mathbb{R}^m)$, where the symbol \rightharpoonup and \rightarrow denote the weak and strong convergence, respectively. We have in fact $u_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$, since $\mathcal{H}(\Omega, \mathbb{R}^m)$ is a closed subspace, hence a weakly closed subset, of $H^1(\Omega, \mathbb{R}^m)$.

For each $j \geq 1$, $|\nabla u_j|^2 + |\nabla u_\varepsilon|^2 \geq 2\nabla u_j \cdot \nabla u_\varepsilon$ in Ω , where the matrix dot-product is defined by $A \cdot B = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$ for all $A = (A_{ij}), B = (B_{ij}) \in \mathbb{R}^{m \times n}$. Thus, by the weak convergence $u_j \rightharpoonup u_\varepsilon$ in $H^1(\Omega, \mathbb{R}^m)$, we have

(2.21)

$$\liminf_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx \geq \liminf_{j \rightarrow \infty} \left[2 \int_{\Omega} \nabla u_j \cdot \nabla u_\varepsilon dx - \int_{\Omega} |\nabla u_\varepsilon|^2 dx \right] = \int_{\Omega} |\nabla u_\varepsilon|^2 dx.$$

By the fact that $\log(1+s) \leq s$ for all $s \geq 0$ and the Cauchy–Schwarz inequality, we imply from the strong convergence $u_j \rightarrow u_\varepsilon$ in $L^2(\Omega, \mathbb{R}^m)$ that

$$\begin{aligned} & \left| \int_{\Omega} [\log(1+|u_j|^2) - \log(1+|u_\varepsilon|^2)] dx \right| = \left| \int_{\Omega} \log \left(1 + \frac{|u_j|^2 - |u_\varepsilon|^2}{1+|u_\varepsilon|^2} \right) dx \right| \\ & \leq \int_{\Omega} \log \left(1 + \left| \frac{|u_j|^2 - |u_\varepsilon|^2}{1+|u_\varepsilon|^2} \right| \right) dx \leq \int_{\Omega} \left| \frac{|u_j|^2 - |u_\varepsilon|^2}{1+|u_\varepsilon|^2} \right| dx \\ (2.22) \quad & \leq (\|u_j\| + \|u_\varepsilon\|) \|u_j - u_\varepsilon\| \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

This and (2.21) thus imply that

$$\mu_\varepsilon = \liminf_{j \rightarrow \infty} I_\varepsilon(u_j) \geq \int_{\Omega} \left[\frac{\varepsilon^2}{2} |\nabla u_\varepsilon|^2 - \frac{1}{2} \log(1+|u_\varepsilon|^2) \right] dx = I_\varepsilon(u_\varepsilon) \geq \mu_\varepsilon,$$

leading to (2.1).

(2) Let $u_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$ be a minimizer of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. The first variation of I_ε at u_ε then vanishes:

$$\delta I_\varepsilon(u_\varepsilon)(v) = \int_{\Omega} \left(\varepsilon^2 \nabla u_\varepsilon \cdot \nabla v - \frac{u_\varepsilon v}{1+u_\varepsilon^2} \right) dx = 0 \quad \forall v \in \mathcal{H}(\Omega, \mathbb{R}^m).$$

Choosing $v = u_\varepsilon$, we obtain that

$$(2.23) \quad \int_{\Omega} |\nabla u_\varepsilon|^2 \leq \frac{1}{\varepsilon^2}.$$

This and the Poincaré inequality (2.18) imply that

$$(2.24) \quad \int_{\Omega} |u_\varepsilon|^2 \leq \frac{C_0^2}{\varepsilon^2}.$$

Since the function $-\log(\cdot)$ is convex, Jensen's inequality and (2.24) then imply that

$$\begin{aligned} (2.25) \quad \mu_\varepsilon = I_\varepsilon(u_\varepsilon) & \geq -\frac{1}{2} \int_{\Omega} \log(1+|u_\varepsilon|^2) dx \geq -\frac{1}{2} \log(1+\|u_\varepsilon\|^2) \\ & \geq -\frac{1}{2} \log \left(1 + \frac{C_0^2}{\varepsilon^2} \right) = -\frac{1}{2} \log(\varepsilon^2 + C_0^2) + \log \varepsilon \geq C_1 + \log \varepsilon, \end{aligned}$$

where $C_1 = -(1/2) \log(1+C_0^2)$.

Let $f : \bar{\Omega} \rightarrow \mathbb{R}$ be the Lipschitz-continuous function constructed in Lemma 2.2. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^m$ be the unit vector along the x_1 -axis. Define $\hat{u}_\varepsilon = (f/\varepsilon)e_1$. Clearly, $\hat{u}_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$. Moreover,

$$(2.26) \quad \mu_\varepsilon \leq I_\varepsilon(\hat{u}_\varepsilon) = \int_{\Omega} \left[\frac{1}{2} |\nabla f|^2 - \frac{1}{2} \log \left(1 + \frac{|f|^2}{\varepsilon^2} \right) \right] dx \leq C_2 + \log \varepsilon,$$

where

$$C_2 := \int_{\Omega} \left(\frac{1}{2} |\nabla f|^2 - \log |f| \right) dx$$

is finite by Lemma 2.2. Now, (2.2) follows from (2.25) and (2.26).

Recall for each $\varepsilon \in (0, 1]$ that the renormalized energy functional, defined in (1.6), is

$$(2.27) \quad \begin{aligned} J_\varepsilon(v) &= I_\varepsilon\left(\frac{v}{\varepsilon}\right) - \log \varepsilon \\ &= \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{2} \log(\varepsilon^2 + |v|^2) \right] dx \quad \forall v \in \mathcal{H}(\Omega, \mathbb{R}^m), \end{aligned}$$

in which the variable v is scaled from the variable v/ε of the energy I_ε . It follows from (1) that for each $\varepsilon \in (0, 1]$ there exists a minimizer of $J_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ and the minimum value of J_ε over $\mathcal{H}(\Omega, \mathbb{R}^m)$ is

$$(2.28) \quad \nu_\varepsilon := \min_{v \in \mathcal{H}(\Omega, \mathbb{R}^m)} J_\varepsilon(v) = \mu_\varepsilon - \log \varepsilon.$$

Consequently, by (2.2), $\{\nu_\varepsilon\}_{0 < \varepsilon \leq 1}$ is bounded. Moreover, for each fixed $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$, we have by (2.27) that $J_\varepsilon(v)$ increases as $\varepsilon \in (0, 1]$ decreases. Therefore, ν_ε increases as $\varepsilon \in (0, 1]$ decreases. This and the boundedness of $\{\nu_\varepsilon\}_{0 < \varepsilon \leq 1}$ imply that $\nu \in \mathbb{R}$ as defined in (2) of Theorem 2.1 is finite and that (2.3) holds true.

(3) Let again $u_\varepsilon \in \mathcal{H}(\Omega, \mathbb{R}^m)$ be a minimizer of $I_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. By (2.24) and (2.23), the upper bound in (2.4) and that in (2.5) hold true with $C_4 = C_0$ and $C_6 = 1$, respectively, for all $\varepsilon \in (0, 1]$. By (2.2) and Jensen's inequality, we obtain

$$C_2 + \log \varepsilon \geq \mu_\varepsilon = I_\varepsilon(u_\varepsilon) \geq -\frac{1}{2} \int_{\Omega} \log(1 + |u_\varepsilon|^2) dx \geq -\frac{1}{2} \log(1 + \|u_\varepsilon\|^2),$$

leading to the lower bound in (2.4) for all $\varepsilon \in (0, e^{-C_2}/\sqrt{2}]$ with $C_3 = e^{-C_2}/\sqrt{2} > 0$. It, together with the Poincaré inequality (2.18), also implies the lower bound in (2.5) for ε in the same range with $C_5 = C_0 C_3 > 0$. Finally, letting $\varepsilon_0 = \min(1, e^{-C_2}/\sqrt{2}) \in (0, 1]$, we obtain all the desired inequalities in (2.4) and (2.5) for all $\varepsilon \in (0, \varepsilon_0]$. \square

Remark 2.1. In the case that $\Omega = \prod_{i=1}^n (a_i, b_i)$ with $-\infty < a_i < b_i < \infty$ ($i = 1, \dots, n$) and the set of admissible functions is $\{u \in H_{\text{per}}^1(\Omega, \mathbb{R}^m) : \int_{\Omega} u dx = 0\}$, the upper bound (2.26) can be obtained by replacing f by $\sin(2\pi x_1/(b_1 - a_1))$.

3. Renormalized energies and their Γ -limit. We consider in this section the convergence of the renormalized energy functionals $J_\varepsilon : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$, defined in (2.27), to the energy functional $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, defined in (1.7).

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}(\Omega, \mathbb{R}^m)$ be the same as in Theorem 2.1. We have*

$$(3.1) \quad -\infty < \inf_{v \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(v) < \infty.$$

Moreover, there exists $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$ such that

$$(3.2) \quad J(v) = \inf_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w).$$

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}(\Omega, \mathbb{R}^m)$ be the same as in Theorem 2.1. Let $\{\varepsilon_j\}_{j=1}^\infty$ be a decreasing sequence in $(0, 1]$ such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. Then, the sequence of functionals $J_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ ($j = 1, \dots$) Γ -converge with respect to the weak topology of $\mathcal{H}(\Omega, \mathbb{R}^m)$ to the functional $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$, i.e., the following hold true:*

(1) *If $v_j \rightharpoonup v$ in $\mathcal{H}(\Omega, \mathbb{R}^m)$, then*

$$(3.3) \quad \liminf_{j \rightarrow \infty} J_{\varepsilon_j}(v_j) \geq J(v);$$

(2) *For any $w \in \mathcal{H}(\Omega, \mathbb{R}^m)$, there exist $w_j \in \mathcal{H}(\Omega, \mathbb{R}^m)$ ($j = 1, \dots$) such that $w_j \rightharpoonup w$ in $\mathcal{H}(\Omega, \mathbb{R}^m)$ and*

$$\lim_{j \rightarrow \infty} J_{\varepsilon_j}(w_j) = J(w).$$

THEOREM 3.3. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}(\Omega, \mathbb{R}^m)$ be the same as in Theorem 2.1. Let $\{\varepsilon_j\}_{j=1}^\infty$ be a decreasing sequence in $(0, 1]$ such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. For each integer $j \geq 1$, let $v_j \in \mathcal{H}(\Omega, \mathbb{R}^m)$ be a minimizer of $J_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. Then, there is a subsequence $\{v_{j_i}\}_{i=1}^\infty$ of $\{v_j\}_{j=1}^\infty$ and $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$ that satisfy the following properties:*

(1) *As $i \rightarrow \infty$, $v_{j_i} \rightarrow v$ (strong convergence) in $\mathcal{H}(\Omega, \mathbb{R}^m)$;*

(2) $J(v) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w)$;

(3) $\lim_{i \rightarrow \infty} \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J_{\varepsilon_{j_i}}(w) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w)$.

COROLLARY 3.1. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}(\Omega, \mathbb{R}^m)$ be the same as in Theorem 2.1.*

(1) *Let $\{\varepsilon_j\}_{j=1}^\infty$ be a decreasing sequence in $(0, 1]$ such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. For each integer $j \geq 1$, let $u_j \in \mathcal{H}(\Omega, \mathbb{R}^m)$ be a minimizer of $I_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. Then, there is a subsequence $\{\varepsilon_{j_i} u_{j_i}\}_{i=1}^\infty$ of $\{\varepsilon_j u_j\}_{j=1}^\infty$ and $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$ that satisfy the following properties: (i) As $i \rightarrow \infty$, $\varepsilon_{j_i} u_{j_i} \rightarrow v$ (strong convergence) in $\mathcal{H}(\Omega, \mathbb{R}^m)$; and (ii) $J(v) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w)$.*

(2) *We have*

$$\lim_{\varepsilon \rightarrow 0^+} \left(\min_{u \in \mathcal{H}(\Omega, \mathbb{R}^m)} I_\varepsilon(u) - \log \varepsilon \right) = \sup_{0 < \varepsilon \leq 1} \left(\min_{u \in \mathcal{H}(\Omega, \mathbb{R}^m)} I_\varepsilon(u) - \log \varepsilon \right) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w).$$

We need several lemmas to prove our results.

LEMMA 3.1. *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable with $0 < |E| < \infty$. Suppose $g_j \rightarrow g$ in $L^1(E)$ and $\{\int_E \log |g_j| dx\}_{j=1}^\infty$ is bounded. Then, $\log |g| \in L^1(E)$ and*

$$(3.4) \quad \liminf_{j \rightarrow \infty} \left(- \int_E \log |g_j| dx \right) \geq - \int_E \log |g| dx.$$

Proof. By the fact that $\log s \leq (1/e)s$ for all $s > 0$, we have for each integer $j \geq 1$ that

$$\begin{aligned}
\int_E |\log |g_j|| \, dx &= \int_{\{x \in E: |g_j(x)| \geq 1\}} \log |g_j| \, dx - \int_{\{x \in E: |g_j(x)| < 1\}} \log |g_j| \, dx \\
&= 2 \int_{\{x \in E: |g_j(x)| \geq 1\}} \log |g_j| \, dx - \int_E \log |g_j| \, dx \\
(3.5) \quad &\leq \frac{2}{e} \int_E |g_j| \, dx - \int_E \log |g_j| \, dx.
\end{aligned}$$

Since both $\{\int_E |g_j| \, dx\}_{j=1}^\infty$ and $\{\int_E \log |g_j| \, dx\}_{j=1}^\infty$ are bounded, we thus have

$$(3.6) \quad \sup_{j \geq 1} \int_E |\log |g_j|| \, dx < \infty.$$

Since $g_j \rightarrow g$ in $L^1(E)$, there exists a subsequence $\{g_{j_i}\}_{i=1}^\infty$ of $\{g_j\}_{j=1}^\infty$ such that $g_{j_i}(x) \rightarrow g(x)$ as $i \rightarrow \infty$ for a.e. $x \in E$. Consequently, by Fatou's Lemma and (3.6),

$$(3.7) \quad 0 \leq \int_E |\log |g|| \, dx = \int_E \liminf_{i \rightarrow \infty} |\log |g_{j_i}|| \, dx \leq \liminf_{i \rightarrow \infty} \int_E |\log |g_{j_i}|| \, dx < \infty.$$

This implies that $\log |g| \in L^1(E)$, and, in particular, $|\{x \in E : g(x) = 0\}| = 0$.

For any $\sigma \in (0, 1)$, we denote $S_\sigma = \{x \in E : 0 < |g(x)| \leq \sigma\}$ and $m_\sigma = |S_\sigma|$. Since $|\log |g|| \geq |\log \sigma|$ on S_σ for any $\sigma \in (0, 1)$, we have by (3.7) that

$$(3.8) \quad m_\sigma = \int_{S_\sigma} dx \leq \int_{S_\sigma} \frac{|\log |g||}{|\log \sigma|} \, dx \leq \frac{1}{|\log \sigma|} \int_E |\log |g|| \, dx \rightarrow 0 \quad \text{as } \sigma \rightarrow 0^+.$$

Thus, by (3.7), (3.8), and the absolute continuity of Lebesgue integrals, we obtain

$$(3.9) \quad \left| \int_{S_\sigma} \log |g| \, dx \right| \leq \int_{S_\sigma} |\log |g|| \, dx \rightarrow 0 \quad \text{as } \sigma \rightarrow 0^+.$$

Now, for each integer $j \geq 1$, we have by the fact that $-\log(\cdot)$ is convex and Jensen's inequality that

$$\begin{aligned}
-\int_{S_\sigma} \log |g_j| \, dx &= -m_\sigma \int_{S_\sigma} \log |g_j| \, dx \geq -m_\sigma \log \left(\int_{S_\sigma} |g_j| \, dx \right) \\
(3.10) \quad &\geq m_\sigma \log m_\sigma - m_\sigma \log \left(\max_{i \geq 1} \|g_i\|_{L^1(E)} \right),
\end{aligned}$$

in which $\max_{i \geq 1} \|g_i\|_{L^1(E)} > 0$, since $g_i \rightarrow g$ in $L^1(E)$ as $i \rightarrow \infty$ and $\|g\|_{L^1(E)} \neq 0$. Thus, by (3.10) and (3.8),

$$\begin{aligned}
\liminf_{j \rightarrow \infty} \left(-\int_{S_\sigma} \log |g_j| \, dx \right) &\geq m_\sigma \log m_\sigma - m_\sigma \log \left(\max_{i \geq 1} \|g_i\|_{L^1(E)} \right) \\
(3.11) \quad &\rightarrow 0 \quad \text{as } \sigma \rightarrow 0^+.
\end{aligned}$$

Let $\delta > 0$. By (3.9) and (3.11), there exists $\sigma_0 \in (0, 1)$ such that

$$(3.12) \quad \liminf_{j \rightarrow \infty} \left(-\int_{S_{\sigma_0}} \log |g_j| \, dx \right) \geq -\int_{S_{\sigma_0}} \log |g| \, dx - \delta.$$

Denoting $T_0 = \{x \in E : |g(x)| > \sigma_0\}$, we have by the fact that $\log(1+s) \leq s$ for any $s \geq 0$ that

$$\begin{aligned}
& \left| \int_{T_0} \log |g_j| dx - \int_{T_0} \log |g| dx \right| \leq \int_{T_0} \left| \log \frac{|g_j|}{|g|} \right| dx \\
& \leq \int_{T_0} \log \left(1 + \frac{|g_j| - |g|}{|g|} \right) dx \leq \int_{T_0} \frac{|g_j - g|}{\sigma_0} dx \\
(3.13) \quad & \leq \frac{1}{\sigma_0} \|g_j - g\|_{L^1(E)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.
\end{aligned}$$

It follows from (3.12) and (3.13) that

$$\liminf_{j \rightarrow \infty} \left(- \int_E \log |g_j| dx \right) \geq - \int_E \log |g| dx - \delta,$$

which implies (3.4) by the arbitrariness of $\delta > 0$. \square

LEMMA 3.2. *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable with $0 < |E| < \infty$ and $h \in L^1(E)$. Let $\{\varepsilon_j\}_{j=1}^\infty$ be a decreasing sequence in $(0, 1]$ such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. Then,*

$$(3.14) \quad \lim_{j \rightarrow \infty} \int_E \log \sqrt{\varepsilon_j^2 + |h|^2} dx = \int_E \log |h| dx.$$

Proof. Suppose first that $\int_E \log |h| dx = -\infty$. Set

$$\zeta_j = \int_E \log \sqrt{\varepsilon_j^2 + |h|^2} dx \quad j = 1, \dots$$

Then, $\{\zeta_j\}_{j=1}^\infty$ is a decreasing sequence. Thus, either $\lim_{j \rightarrow \infty} \zeta_j = -\infty$, leading to (3.14) in this case; or $\lim_{j \rightarrow \infty} \zeta_j$ exists and is finite. Suppose the latter were true. Then, $\{\zeta_j\}_{j=1}^\infty$ would be bounded from below. By the fact that $\log s \leq (1/e)s$ for any $s > 0$, we have for any $j \geq 1$ that

$$(3.15) \quad \zeta_j \leq \int_E \log \sqrt{1 + |h|^2} dx \leq \frac{1}{e} \int_E \sqrt{1 + |h|^2} dx \leq \frac{1}{e} \int_E (1 + |h|) dx < \infty.$$

Thus, the sequence $\{\zeta_j\}_{j=1}^\infty$ is also bounded from above. In addition, $\sqrt{\varepsilon_j^2 + |h|^2} \rightarrow |h|$ in $L^1(E)$ as $j \rightarrow \infty$. Therefore, by Lemma 3.1, $\int_E \log |h| dx$ would be finite, leading to a contradiction in this case.

Suppose now that $\int_E \log |h| dx > -\infty$. Replacing g_j by h in (3.5), we obtain that $\log |h| \in L^1(E)$. By (3.15), $\log \sqrt{1 + |h|^2} \in L^1(E)$. Since for each $j \geq 1$,

$$\log |h| \leq \log \sqrt{\varepsilon_j^2 + |h|^2} \leq \log \sqrt{1 + |h|^2} \quad \text{a.e. } E,$$

we thus obtain (3.14) in this case by Lebesgue's Dominated Convergence Theorem. \square

Proof of Theorem 3.1. Let $\tau = \inf_{v \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(v)$. Let $f : \bar{\Omega} \rightarrow \mathbb{R}$ be the Lipschitz-continuous function constructed in Lemma 2.2. Define $\hat{v} : \bar{\Omega} \rightarrow \mathbb{R}^m$ by $\hat{v} = f e_1$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^m$. Then, $\hat{v} \in \mathcal{H}(\Omega, \mathbb{R}^m)$ and $J(\hat{v}) < \infty$. Thus, $\tau < \infty$.

Since $(1/s) \log s \rightarrow 0$ as $s \rightarrow \infty$, there exists $R = R(\Omega) > 1$ such that

$$(3.16) \quad \log s \leq \frac{s}{2C_0^2} \quad \forall s \geq R,$$

where $C_0 > 0$ is the constant in the Poincaré inequality (2.18). Consequently, by (3.16) and the Poincaré inequality (2.18),

$$(3.17) \quad \begin{aligned} J(v) &\geq \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{|\Omega|} \int_{\{x \in \Omega: |v| \geq 1\}} \log |v| dx \\ &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: 1 \leq |v|^2 \leq R\}} \log(|v|^2) dx \\ &\quad - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: |v|^2 > R\}} \log(|v|^2) dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{2|\Omega|} \int_{\{x \in \Omega: 1 \leq |v|^2 \leq R\}} \log R dx - \frac{1}{4|\Omega|C_0^2} \int_{\{x \in \Omega: |v|^2 > R\}} |v|^2 dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{4C_0^2} \int_{\Omega} |v|^2 dx - \frac{1}{2} \log R \\ &\geq \frac{1}{4} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{2} \log R \quad \forall v \in \mathcal{H}(\Omega, \mathbb{R}^m). \end{aligned}$$

This implies that $\tau > -\infty$. Hence, (3.1) is proved.

Let $\{v_j\}_{j=1}^{\infty}$ be an infimizing sequence of $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$. It follows from (3.17) and (2.18) that $\{v_j\}_{j=1}^{\infty}$ is bounded in $\mathcal{H}(\Omega, \mathbb{R}^m)$. Thus, up to a subsequence, $v_j \rightharpoonup v$ in $H^1(\Omega, \mathbb{R}^m)$ and $v_j \rightarrow v$ in $L^2(\Omega, \mathbb{R}^m)$ as $j \rightarrow \infty$ for some $v \in H^1(\Omega, \mathbb{R}^m)$. We have $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$, since $\mathcal{H}(\Omega, \mathbb{R}^m)$ is weakly closed in $H^1(\Omega, \mathbb{R}^m)$.

As in the proof of Theorem 2.1, cf. (2.21), we have

$$(3.18) \quad \liminf_{j \rightarrow \infty} \int_{\Omega} |\nabla v_j|^2 dx \geq \int_{\Omega} |\nabla v|^2 dx.$$

Since $\{J(v_j)\}_{j=1}^{\infty}$ and $\{\|v_j\|\}_{j=1}^{\infty}$ are both bounded, the sequence

$$\left\{ \int_{\Omega} \log |v_j| dx \right\}_{j=1}^{\infty} = \left\{ \int_{\Omega} \frac{1}{2} |\nabla v_j|^2 dx - |\Omega| J(v_j) \right\}_{j=1}^{\infty}$$

is bounded. Thus, by Lemma 3.1, $\log |v| \in L^1(\Omega)$ and

$$(3.19) \quad \liminf_{j \rightarrow \infty} \left(- \int_{\Omega} \log |v_j| dx \right) \geq - \int_{\Omega} \log |v| dx.$$

Now, (3.2) follows from (3.18), (3.19), and the fact that $\{v_j\}_{j=1}^{\infty}$ is an infimizing sequence of $J : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{\infty\}$. \square

Remark 3.1. In the case that $\Omega = \prod_{i=1}^n (a_i, b_i)$ with $-\infty < a_i < b_i < \infty$ ($i = 1, \dots, n$) and the set of admissible functions is $\{u \in H_{\text{per}}^1(\Omega, \mathbb{R}^m) : \int_{\Omega} u dx = 0\}$, we can still prove that $\tau < \infty$ by the same argument with f replaced by $\sin(2\pi x_1 / (b_1 - a_1))$.

Proof of Theorem 3.2.

(1) Suppose $v_j \rightharpoonup v$ in $\mathcal{H}(\Omega, \mathbb{R}^m)$. We may assume that $\liminf_{j \rightarrow \infty} J_{\varepsilon_j}(v_j) < \infty$, otherwise (3.3) holds true trivially.

Notice that for each integer $j \geq 1$,

$$(3.20) \quad J_{\varepsilon_j}(v_j) \geq \int_{\Omega} \left[\frac{1}{2} |v_j|^2 - \frac{1}{2} \log(1 + |v_j|^2) \right] dx = I_1(v_j).$$

Thus, by (2.20) with $\varepsilon = 1$, the sequence $\{J_{\varepsilon_j}(v_j)\}_{j=1}^{\infty}$ is bounded from below. Let $\{v_{j_i}\}_{i=1}^{\infty}$ be a subsequence of $\{v_j\}_{j=1}^{\infty}$ such that $v_{j_i} \rightarrow v$ in $L^2(\Omega, \mathbb{R}^m)$ as $i \rightarrow \infty$ and

$$(3.21) \quad \liminf_{j \rightarrow \infty} J_{\varepsilon_j}(v_j) = \lim_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(v_{j_i}) < \infty.$$

Then, the sequence $\{J_{\varepsilon_{j_i}}(v_{j_i})\}_{i=1}^{\infty}$ is bounded, from both above and below. Hence,

$$\left\{ \int_{\Omega} \log \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} dx \right\}_{i=1}^{\infty} = \left\{ \int_{\Omega} \frac{1}{2} |\nabla v_{j_i}|^2 dx - |\Omega| J_{\varepsilon_{j_i}}(v_{j_i}) \right\}_{i=1}^{\infty}$$

is bounded. Moreover, $\sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} \rightarrow |v|$ in $L^2(\Omega)$ as $i \rightarrow \infty$, since

$$\left| \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} - |v| \right|^2 = \varepsilon_{j_i}^2 + |v_{j_i}|^2 + |v|^2 - 2|v| \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} \leq \varepsilon_{j_i}^2 + |v_{j_i} - v|^2 \quad \forall i \geq 1.$$

Consequently, it follows from Lemma 3.1 that $\log |v| \in L^1(\Omega)$ and

$$(3.22) \quad \liminf_{i \rightarrow \infty} \left(- \int_{\Omega} \log \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} dx \right) \geq - \int_{\Omega} \log |v| dx.$$

As before, we also have

$$(3.23) \quad \liminf_{j \rightarrow \infty} \int_{\Omega} |\nabla v_j|^2 dx \geq \int_{\Omega} |\nabla v|^2 dx,$$

cf. (2.21) and (3.18). Now, (3.3) follows from (3.21)–(3.23).

(2) Let $w \in \mathcal{H}(\Omega, \mathbb{R}^m)$ and $w_j = w$ for all integers $j \geq 1$. The assertion of this part follows from Lemma 3.2. \square

Proof of Theorem 3.3. For each integer $j \geq 1$, $\varepsilon_j v_j$ is a minimizer of $I_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. Thus, by (3) of Theorem 2.1, $\{v_j\}_{j=1}^{\infty}$ is bounded in $\mathcal{H}(\Omega, \mathbb{R}^m)$. Hence, it has a subsequence $\{v_{j_i}\}_{i=1}^{\infty}$ such that $v_{j_i} \rightharpoonup v$ in $\mathcal{H}(\Omega, \mathbb{R}^m)$, $v_{j_i} \rightarrow v$ in $L^2(\Omega, \mathbb{R}^m)$, and $v_{j_i}(x) \rightarrow v(x)$ for a.e. $x \in \Omega$ as $i \rightarrow \infty$ for some $v \in \mathcal{H}(\Omega, \mathbb{R}^m)$.

Since v_{j_i} is a minimizer of $J_{\varepsilon_{j_i}} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$, we have for any $w' \in \mathcal{H}(\Omega, \mathbb{R}^m)$ that

$$J_{\varepsilon_{j_i}}(w') \geq J_{\varepsilon_{j_i}}(v_{j_i}) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J_{\varepsilon_{j_i}}(w) \quad \forall i \geq 1.$$

Consequently, we have by Lemma 3.2 and (1) of Theorem 3.2 that

$$(3.24) \quad J(w') = \lim_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(w') \geq \limsup_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(v_{j_i}) \geq \liminf_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(v_{j_i}) \geq J(v).$$

This proves (2). Setting $w' = v$ in (3.24), we obtain that

$$(3.25) \quad \lim_{i \rightarrow \infty} \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J_{\varepsilon_{j_i}}(w) = \lim_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(v_{j_i}) = J(v) = \min_{w \in \mathcal{H}(\Omega, \mathbb{R}^m)} J(w),$$

proving (3).

Notice that the sequence

$$\left\{ \int_{\Omega} \log \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} dx \right\}_{i=1}^{\infty} = \left\{ \int_{\Omega} \frac{1}{2} |\nabla v_{j_i}|^2 dx - |\Omega| J_{\varepsilon_{j_i}}(v_{j_i}) \right\}_{i=1}^{\infty}$$

is bounded. Moreover, $\sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} \rightarrow |v|$ in $L^2(\Omega)$ as $i \rightarrow \infty$. Consequently, we have by Lemma 3.1 that $\log |v| \in L^1(\Omega)$ and

$$(3.26) \quad \liminf_{i \rightarrow \infty} \left(- \int_{\Omega} \log \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} dx \right) \geq - \int_{\Omega} \log |v| dx.$$

Since $v_{j_i} \rightarrow v$ in $\mathcal{H}(\Omega, \mathbb{R}^m)$ as $i \rightarrow \infty$, we also have (cf. (2.21))

$$(3.27) \quad \liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla v_{j_i}|^2 dx \geq \int_{\Omega} |\nabla v|^2 dx.$$

Now, it follows from (3.25)–(3.27) that

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} J_{\varepsilon_{j_i}}(v_{j_i}) - J(v) \\ &\geq \left[\liminf_{i \rightarrow \infty} \int_{\Omega} \frac{1}{2} |\nabla v_{j_i}|^2 dx - \int_{\Omega} \frac{1}{2} |\nabla v|^2 dx \right] \\ &\quad + \left[\liminf_{i \rightarrow \infty} \left(- \int_{\Omega} \log \sqrt{\varepsilon_{j_i}^2 + |v_{j_i}|^2} dx \right) - \left(- \int_{\Omega} \log |v| dx \right) \right] \\ &\geq 0, \end{aligned}$$

which, together with (3.26) and (3.27), implies that

$$\liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla v_{j_i}|^2 dx = \int_{\Omega} |\nabla v|^2 dx.$$

Thus,

$$\liminf_{i \rightarrow \infty} \int_{\Omega} |\nabla v_{j_i} - \nabla v|^2 dx = \liminf_{i \rightarrow \infty} \int_{\Omega} (|\nabla v_{j_i}|^2 + |\nabla v|^2 - 2\nabla v_{j_i} \cdot \nabla v) dx = 0.$$

This and the Poincaré inequality (2.18) imply the strong convergence of a subsequence of $\{v_{j_i}\}_{i=1}^{\infty}$ to v in $\mathcal{H}(\Omega, \mathbb{R}^m)$. Thus (1) is proved. \square

Proof of Corollary 3.1. Notice for any integer $j \geq 1$ that u_j is a minimizer of $I_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$ if and only if that $v_j := \varepsilon_j u_j$ is a minimizer of $J_{\varepsilon_j} : \mathcal{H}(\Omega, \mathbb{R}^m) \rightarrow \mathbb{R}$. Thus, (1) follows from (1) and (2) of Theorem 3.3; (3) follows from (2) of Theorem 2.1 and (3) of Theorem 3.3. \square

4. The scalar Dirichlet boundary-value problem. If the order parameter is a scalar function that satisfies the homogeneous Dirichlet boundary condition, then the solution of the corresponding limiting variational problem of infimizing $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ can be well characterized. In what follows, we denote

$$\mathcal{H}_+(\Omega) = \{v \in H_0^1(\Omega) : v(x) \geq 0 \text{ a.e. } x \in \Omega\}.$$

As usual, we also denote by $C_c^\infty(\Omega)$ the set of all $C^\infty(\Omega)$ -functions that are compactly supported in Ω .

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. Then, there exists $v_+ \in \mathcal{H}_+(\Omega)$ that satisfies the following properties:*

(1) *The function v_+ is the unique minimizer of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. Moreover, $v_+ \in C^\infty(\Omega)$, $v_+ > 0$ in Ω ,*

$$(4.1) \quad \Delta v_+ + \frac{1}{v_+} = 0 \quad \text{in } \Omega,$$

and

$$(4.2) \quad \int_{\Omega} |\nabla v_+|^2 dx = 1;$$

(2) *The two functions v_+ and $v_- := -v_+$ are the unique minimizers of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$.*

To prove this theorem, we need the following result.

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^n$ be the same as in Theorem 4.1. Let $\varepsilon \in (0, 1]$. There exists $v_{\varepsilon+} \in \mathcal{H}_+(\Omega)$ such that*

$$(4.3) \quad J_\varepsilon(v_{\varepsilon+}) = \min_{w \in \mathcal{H}_+(\Omega)} J_\varepsilon(w) = \min_{w \in H_0^1(\Omega)} J_\varepsilon(w).$$

Moreover, $\Delta v_{\varepsilon+} \leq 0$ in Ω in the sense of distributions, i.e.,

$$(4.4) \quad \int_{\Omega} \nabla v_{\varepsilon+} \cdot \nabla \varphi dx \geq 0 \quad \forall \varphi \in C_c^\infty(\Omega) \text{ with } \varphi \geq 0 \text{ in } \Omega,$$

and

$$(4.5) \quad \int_{\Omega} |\nabla v_{\varepsilon+}|^2 dx = \int_{\Omega} \frac{v_{\varepsilon+}^2}{\varepsilon^2 + v_{\varepsilon+}^2} dx.$$

Proof. Setting $\xi_\varepsilon = \inf_{w \in \mathcal{H}_+(\Omega)} J_\varepsilon(w)$, we have by Theorem 2.1 that

$$-\infty < \min_{w \in H_0^1(\Omega)} I_\varepsilon(w) - \log \varepsilon = \min_{w \in H_0^1(\Omega)} J_\varepsilon(w) \leq \xi_\varepsilon \leq J_\varepsilon(0) = -\log \varepsilon < \infty.$$

Let $\{w_j\}_{j=1}^\infty \subset \mathcal{H}_+(\Omega)$ be an infimizing sequence of $J_\varepsilon : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R}$. Since

$$J_\varepsilon(w_j) \geq \int_{\Omega} \left[\frac{1}{2} |\nabla w_j|^2 - \frac{1}{2} \log(1 + |w_j|^2) \right] dx = I_1(w_j) \quad \forall j \geq 1,$$

we see from (2.20) with $\mathbb{R}^m = \mathbb{R}$ and $\varepsilon = 1$, and the Poincaré inequality, that the sequence $\{w_j\}_{j=1}^\infty$ is bounded in $H^1(\Omega)$. Thus, it has a subsequence $\{w_{j_i}\}_{i=1}^\infty$ such that $w_{j_i} \rightharpoonup v_{\varepsilon+}$ in $H^1(\Omega)$ and $w_{j_i} \rightarrow v_{\varepsilon+}$ in $L^2(\Omega)$ as $i \rightarrow \infty$ for some $v_{\varepsilon+} \in H^1(\Omega)$. We have in fact $v_+ \in \mathcal{H}_+(\Omega)$, since $\mathcal{H}_+(\Omega)$ is convex and strongly closed, and hence weakly closed, in $H^1(\Omega)$. Noting that $\varepsilon > 0$ is fixed, by the same argument in the proof of Theorem 2.1, cf. (2.21) and (2.22), we obtain that $J_\varepsilon(v_{\varepsilon+}) = \xi_\varepsilon$.

For any $w \in H_0^1(\Omega)$, we have $|w| \in \mathcal{H}_+(\Omega)$ and $J_\varepsilon(|w|) = J_\varepsilon(w)$, cf. Lemmas 7.6 and 7.7 in [8]. Thus,

$$\min_{w \in H_0^1(\Omega)} J_\varepsilon(w) \leq \xi_\varepsilon = J_\varepsilon(v_{\varepsilon+}) \leq J_\varepsilon(|w|) = J_\varepsilon(w) \quad \forall w \in H_0^1(\Omega).$$

This leads to (4.3).

Since $v_{\varepsilon+}$ is a minimizer of $J_\varepsilon : H_0^1(\Omega) \rightarrow \mathbb{R}$, the first variation of J_ε at $v_{\varepsilon+}$ vanishes, i.e.,

$$(4.6) \quad \delta J(v_{\varepsilon+})(\varphi) = \int_{\Omega} \left(\nabla v_{\varepsilon+} \cdot \nabla \varphi - \frac{v_{\varepsilon+} \varphi}{\varepsilon^2 + v_{\varepsilon+}^2} \right) dx = 0 \quad \forall \varphi \in H_0^1(\Omega).$$

This, together with the fact that $v_{\varepsilon+} \geq 0$ a.e. Ω and that $C_c^\infty(\Omega) \subset H_0^1(\Omega)$, implies (4.4). Finally, setting $\varphi = v_{\varepsilon+}$ in (4.6), we obtain (4.5). \square

Proof of Theorem 4.1.

(1) By Lemma 4.1, there exists $v_j \in \mathcal{H}_+(\Omega)$ for each integer $j \geq 1$ such that

$$(4.7) \quad J_{1/j}(v_j) = \min_{w \in \mathcal{H}_+(\Omega)} J_{1/j}(w) = \min_{w \in H_0^1(\Omega)} J_{1/j}(w),$$

$$(4.8) \quad \Delta v_j \leq 0 \quad \text{in } \Omega$$

in the sense of distributions, cf. (4.4), and

$$(4.9) \quad \int_{\Omega} |\nabla v_j|^2 dx = \int_{\Omega} \frac{v_j^2}{j^{-2} + v_j^2} dx.$$

By (4.7) and Theorem 3.3, there exists a subsequence $\{v_{j_i}\}_{i=1}^\infty$ of $\{v_j\}_{j=1}^\infty$ such that $v_{j_i} \rightarrow v_+$ (strong convergence) in $H^1(\Omega)$ and $v_{j_i}(x) \rightarrow v_+(x)$ a.e. $x \in \Omega$ as $i \rightarrow \infty$ for some $v_+ \in H_0^1(\Omega)$ that is a minimizer of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. Since $\mathcal{H}_+(\Omega)$ is weakly closed in $H^1(\Omega)$, $v_+ \in \mathcal{H}_+(\Omega)$. Moreover, since $\mathcal{H}_+(\Omega) \subset H_0^1(\Omega)$, v_+ is also a minimizer of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. The fact that v_+ is the unique minimizer of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ follows from the strict convexity of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$.

Let $\eta_+ = \text{ess inf}_\Omega v_+$. Since $v_+ \in \mathcal{H}_+(\Omega)$, $v_+ \geq 0$ a.e. Ω . Thus, $\eta_+ \geq 0$. If $\eta_+ > 0$, then there exists $\phi_j \in C^\infty(\bar{\Omega})$ for each integer $j \geq 1$ such that $\min_{\bar{\Omega}} \phi_j \geq \eta_+/2$ ($j = 1, \dots$) and $\phi_j \rightarrow v_+$ in $H^1(\Omega)$ as $j \rightarrow \infty$. The trace of v_+ , which is the limit of $\{\phi_j\}_{j=1}^\infty$ in $L^2(\partial\Omega)$, would then be positive a.e. $\partial\Omega$. This contradicts the fact that $v_+ \in H_0^1(\Omega)$. Thus, $\text{ess inf}_\Omega v_+ = 0$. Since $v_+ \geq 0$ is a minimizer of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, $\log v_+ \in L^1(\Omega)$. Thus, we also have that $v_+(x) > 0$ a.e. $x \in \Omega$. In particular, v_+ is not a constant in Ω . Since $v_{j_i} \rightarrow v_+$ in $H^1(\Omega)$ as $i \rightarrow \infty$, we obtain by (4.8) that $\Delta v_+ \geq 0$ in Ω in the sense of distributions. Applying the Strong Maximum Principle to $L = \Delta$ and $u = v_+$ in Theorem 8.19 in [8], we see that $\text{ess inf}_B v_+ > 0$ for any ball $B \subset\subset \Omega$. (Here and below, the notation $\omega \subset\subset \Omega$ means $\bar{\omega} \subset \Omega$.) For any open set $\Omega' \subset \mathbb{R}^n$ with $\Omega' \subset\subset \Omega$, we can cover $\bar{\Omega}'$ by finitely many balls $B_i \subset\subset \Omega$, where $i = 1, \dots, N$ for some integer $N \geq 1$, so that $\text{ess inf}_{B_i} v_+ > 0$ for all i ($1 \leq i \leq N$). Thus, $\delta' := \text{ess inf}_{\Omega'} v_+ > 0$, and hence $1/v_+ \in L^\infty(\Omega')$.

Let $\varphi \in C_c^\infty(\Omega)$ with $\text{supp } \varphi \subset \Omega'$ and consider $q(\delta) := J(v_+ + \delta\varphi)$ for $\delta \in \mathbb{R}$. If $|\delta| \text{sup}_{\Omega'} |\varphi| < \delta'$, then $v_+ + \delta\varphi > 0$ a.e. in Ω' , and

$$\begin{aligned} q(\delta) &= \int_{\Omega} \left[\frac{1}{2} |\nabla v_+ + \delta\varphi|^2 - \log |v_+ + \delta\varphi| \right] dx \\ &= \int_{\Omega} \frac{1}{2} |\nabla v_+ + \delta\varphi|^2 dx - \frac{1}{|\Omega|} \int_{\Omega \setminus \Omega'} \log v_+ dx - \frac{1}{|\Omega|} \int_{\Omega'} \log(v_+ + \delta\varphi) dx. \end{aligned}$$

Since v_+ is a minimizer of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, $q(\delta)$ is minimized at $\delta = 0$. Thus, $q'(0) = 0$. This leads to

$$(4.10) \quad \Delta v_+ + \frac{1}{v_+} = 0 \quad \text{in } \Omega'$$

in the sense of distributions.

Let $\Omega'' \subset \mathbb{R}^n$ be an open set such that $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Denoting $H^k(D) = W^{k,2}(D)$ the usual Sobolev space for an open set $D \subset \mathbb{R}^n$ and an integer $k \geq 1$ [1, 8], we claim:

(*) For any integer $k \geq 1$, there exists an open set $\Omega_k \subset \mathbb{R}^n$ such that $\Omega'' \subset\subset \Omega_k \subset\subset \Omega'$ and $v_+ \in H^{k+1}(\Omega_k)$.

Let $f := 1/v_+ \in L^\infty(\Omega')$. By (4.10), $v_+ \in H^1(\Omega')$ is a weak solution of $\Delta v_+ = f$ in Ω' , i.e.,

$$(4.11) \quad \int_{\Omega'} \nabla v_+ \cdot \nabla \varphi \, dx = \int_{\Omega'} f \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega').$$

By the regularity theory of elliptic boundary-value problems, cf. Theorem 8.8 in [8], we have $v_+ \in H^2(\Omega_1)$ for any open set $\Omega_1 \subset \mathbb{R}^n$ such that $\Omega'' \subset\subset \Omega_1 \subset\subset \Omega'$. Thus, the statement (*) is true for $k = 1$.

Suppose the statement (*) is also true for a general $k \geq 1$. Then, $\partial^k f \in H^1(\Omega_k)$ for any partial derivative ∂^k of order k . Replacing φ in (4.11) by $\partial^k \psi$ for any $\psi \in C_c^\infty(\Omega_k)$, one easily verifies that

$$\int_{\Omega_k} \nabla \partial^k v_+ \cdot \nabla \psi \, dx = \int_{\Omega_k} \partial^k f \psi \, dx \quad \forall \psi \in C_c^\infty(\Omega_k),$$

i.e., $\partial^k v_+ \in H^1(\Omega_k)$ satisfies $\Delta \partial^k v_+ = \partial^k f$ in Ω_k in the sense of distributions. Therefore, by the same regularity result, there exists an open set $\Omega_{k+1} \subset \mathbb{R}^n$ with $\Omega'' \subset\subset \Omega_{k+1} \subset\subset \Omega_k \subset\subset \Omega'$ such that $\partial^k v_+ \in H^2(\Omega_{k+1})$. By suitably enlarging Ω_{k+1} if necessary, we see that $v_+ \in H^{k+2}(\Omega_{k+1})$. Hence, the statement (*) is true for $k + 1$. Thus, it is true for any integer $k \geq 1$.

By the statement (*), $v_+ \in C^\infty(\Omega'')$. It then follows from the arbitrariness of Ω'' and Ω' that $v_+ \in C^\infty(\Omega)$, and that $v_+ > 0$ in Ω , since $\text{ess sup}_{\Omega'} v_+ > 0$ for any $\Omega' \subset\subset \Omega$. Moreover, (4.1) follows from (4.10) and the arbitrariness of $\Omega' \subset\subset \Omega$. Using the fact that $v_{j_i} \rightarrow v_+$ in $H^1(\Omega)$ and that $v_{j_i}(x) \rightarrow v_+(x) > 0$ a.e. Ω , and applying Lebesgue's Dominated Convergence Theorem, we obtain (4.2) from (4.9).

(2) Clearly, both v_+ and v_- are minimizers of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, cf. the proof of Lemma 4.1. Assume now $\tilde{v} \in H_0^1(\Omega)$ is a minimizer of $J : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. Then, $|\tilde{v}| \in \mathcal{H}_+(\Omega)$ is a minimizer of $J : \mathcal{H}_+(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. By (1), we must have that $|\tilde{v}| = v_+$ a.e. Ω . Thus, $|\tilde{v}| \in C^\infty(\Omega)$ and $\tilde{v} > 0$ in Ω . Consequently, we have for any ball $B \subset\subset \Omega$ that $\tilde{v}(x) > 0$ for all $x \in B$ or $\tilde{v}(x) < 0$ for all $x \in B$. Therefore, for any domain $\omega \subset\subset \Omega$, $\tilde{v}(x)$ has the same sign for each $x \in \bar{\omega}$. This implies that $\tilde{v}(x) = v_+(x)$ for all $x \in \Omega$ or $\tilde{v}(x) = v_-(x)$ for all $x \in \Omega$. \square

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REFERENCES

- [1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] J. W. CAHN AND J. E. HILLIARD, *Free energy of a nonuniform system I. Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258–267.
- [3] M. CYROT, *Ginzburg-Landau theory for superconductors*, Rep. Progr. Phys., 36 (1973), pp. 103–158.
- [4] G. EHRLICH AND F. G. HUDDA, *Atomic view of surface diffusion: Tungsten on tungsten*, J. Chem. Phys., 44 (1966), p. 1036.
- [5] H.-J. ERNST, F. FABRE, R. FOLKERTS, AND J. LAPUJOLADE, *Observation of a growth instability during low temperature molecular beam epitaxy*, Phys. Rev. Lett., 72 (1994), pp. 112–115.

- [6] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.
- [7] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
- [8] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Grundlehren Math. Wiss., Springer-Verlag, Berlin, 1998.
- [9] V. L. GINZBURG AND L. D. LANDAU, *Zh. Eksp. Teor. Fiz.*, 20 (1950), p. 1064.
- [10] L. GOLUBOVIĆ, *Interfacial coarsening in epitaxial growth models without slope selection*, *Phys. Rev. Lett.*, 78 (1997), pp. 90–93.
- [11] A. W. HUNT, C. ORME, D. R. M. WILLIAMS, B. G. ORR, AND L. M. SANDER, *Instabilities in MBE growth*, *Europhys. Lett.*, 27 (1994), p. 611.
- [12] M. D. JOHNSON, C. ORME, A. W. HUNT, D. GRAFF, J. SUDIJONO, L. M. SANDER, AND B. G. ORR, *Stable and unstable growth in molecular beam epitaxy*, *Phys. Rev. Lett.*, 72 (1994), pp. 116–119.
- [13] B. LI AND J.-G. LIU, *Epitaxial growth without slope selection: Energetics, coarsening, and dynamic scaling*, *J. Nonlinear Sci.*, 14 (2004), pp. 429–451.
- [14] S. OSHER AND R. FEDKIW, *Level Set Methods and Dynamic Implicit Surfaces*, Springer, New York, 2002.
- [15] P. POLITI AND A. TORCINI, *Coarsening in surface growth models without slope selection*, *J. Phys. A*, 33 (2000), pp. L77–L82.
- [16] M. ROST AND J. KRUG, *Coarsening of surface structures in unstable epitaxial growth*, *Phys. Rev. E*, 55 (1997), pp. 3952–3957.
- [17] R. L. SCHWOEBEL AND E. J. SHIPSEY, *Step motion on crystal surfaces*, *J. Appl. Phys.*, 37 (1966), p. 3682.
- [18] R. L. SCHWOEBEL, *Step motion on crystal surfaces II*, *J. Appl. Phys.*, 40 (1969), p. 614.
- [19] J. A. SETHIAN, *Level Set Methods and Fast Marching Methods*, Cambridge University Press, 1999.
- [20] K. THÜRMER, R. KOCH, M. WEBER, AND K. H. RIEDER, *Dynamic evolution of pyramid structures during growth of epitaxial Fe(001) films*, *Phys. Rev. Lett.*, 75 (1995), pp. 1767–1770.
- [21] J. VILLAIN, *Continuum models of crystal growth from atomic beams with and without desorption*, *J. de Phys. I*, 1 (1991), pp. 19–42.