High-Order Surface Relaxation vs. the Ehrlich-Schwoebel Effect

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September 6, 2006

Abstract

We consider a class of continuum models of epitaxial growth of thin films with two competing mechanisms: (1) the surface relaxation described by high-order gradients of the surface profile; and (2) the Ehrlich-Schwoebel (ES) effect which is the asymmetry in the adatom attachment and detachment to and from atomic steps. Mathematically, these models are gradient-flows of some effective free-energy functionals for which large slopes are preferred for surfaces with low energy.

We characterize the large-system asymptotics of the minimum energy and the magnitude of gradients of energy-minimizing surfaces. We also show that, in the large-system limit, the renormalized energy with an infinite ES barrier is the Γ-limit of those with a finite one, indicating the enhancement of the ES effect in a large system. Introducing λ-minimizers as energy minimizers among all candidates that are spatially λ-periodical, we show the existence of a sequence of such λ-minimizers that are in fact equilibriums. For the case of a finite ES effect, we prove the well-posedness of the initial-boundary-value problem of the continuum model; and obtain bounds for the scaling laws of interface width, surface slope, and energy, all of which characterize the surface coarsening during the film growth. We conclude with a discussion on implications of our rigorous analysis.

MSC: 34K26, 49J45, 74G65, 74K30.

PACS: 68.35.Ct; 68.43.Jk; 81.15.Aa.

Key words and phrases: epitaxial growth of thin films, surface relaxation, the Ehrlich-Schwoebel effect, energy minimization, Γ-convergence, coarsening, scaling laws.

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1 Introduction

In the form of mass balance, a continuum model of epitaxial growth of thin films is given by

$$\partial_t h + \nabla \cdot \mathbf{j} = F, \tag{1.1}$$

where $h = h(x,t)$ with $x = (x_1, x_2)$ is the coarse-grained height profile of the film surface at time t, $\mathbf{j} = \mathbf{j}(\nabla h, \nabla^2 h, \ldots)$ is the surface current or flux which depends only on gradients of h but not explicitly on x due to the translational invariance in h and x, and F is the mean deposition flux which we assume to be a positive constant [4, 24]. Here, we neglect the noise in the deposition flux.

The current $\mathbf{j}$ describes microscopic processes in the growth that determine macroscopic properties of films and growth scaling laws. It can include many different processes and mechanisms. In this work, we consider two important such processes and mechanisms that compete each other.

The first one is a high-order surface relaxation, i.e., surface relaxation described by high-order gradients of the height profile h. This process smoothens the surface in general. The current due to such relaxation is given by

$$\mathbf{j}_{RE} = (-1)^m M_m \nabla \Delta^{m-1} h, \tag{1.2}$$

where $m \geq 2$ is an integer, M_m the mobility which is taken to be a positive constant here, ∇ the gradient, and Δ the Laplacian. Here, we assume that the surface current is isotropic—often, an idealized situation in the growth of a crystalline surface.

For $m = 2$, the current (1.2) is the Herring-Mullins term for the isotropic surface diffusion [11, 23]. It can be derived for some cases from a Burton-Cabrera-Frank (BCF) type model [6, 15, 22]. The surface relaxation with $m = 3$ in (1.2), together with lower-order terms, has been suggested to model the homoepitaxy of Fe(001) at room temperature [33]. The physical origin of the relaxation with $m \geq 3$ remains unclear and a satisfactory derivation of such a term seems to be challenging.

The second one is the adatom (adsorbed atom) attachment-detachment with the Ehrlich-Schwoebel (ES) effect: in order to stick to an atomic step, an adatom from an *upper* terrace must overcome an energy barrier—the ES barrier—in addition to the diffusion barrier on an atomistically flat terrace [7, 29, 30]; cf. Figure 1.

The ES effect generates an uphill current that destabilizes nominal surfaces (high-symmetry surfaces), but stabilizes vicinal surfaces (stepped surfaces that are in the vicinity of high-symmetry surfaces) with a large slope, preventing step bunching [7, 29, 36]. It is the origin of the Bales-Zangwill instability, a diffusional instability of atomic steps [3]. And it also affects the island nucleation [17]. With the ES effect, the film surface prefers large slope. The competition between this large-slope preference and the surface relaxation determines the large-scale surface morphology and growth scaling laws [2, 10, 12, 20, 25, 27, 31, 36].

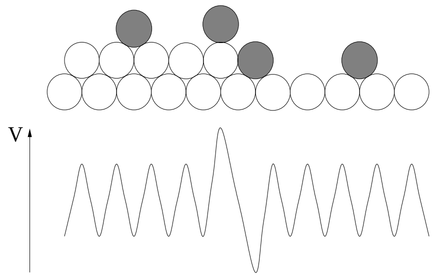

Figure 1. The Ehrlich-Schwoebel barrier.

The exact form of the current induced by an infinite ES barrier was first proposed in [36] and that by a finite ES barrier in [12]; see also [16, 26, 31]. These forms are:

$$\mathbf{j}_{ES} = \begin{cases} \dfrac{F}{2}\dfrac{\nabla h}{|\nabla h|^2} & \text{for an infinite ES barrier,} \\ \dfrac{F S_c \sigma^2 \nabla h}{1+\alpha^2\sigma^2|\nabla h|^2} & \text{for a finite ES barrier,} \end{cases} \tag{1.3}$$

where $S_c > 0$ is a constant measuring the strength of ES effect, $\sigma > 0$ the nucleation length, and $\alpha > 0$ an interpolation constant—a fitting parameter [12]. For the case of an infinite ES barrier, the current is also given in [12] in a slightly altered form (without the factor 1/2).

Now, the total surface current $\mathbf{j}$ is the sum of $\mathbf{j}_{RE}$ and $\mathbf{j}_{ES}$:

$$\mathbf{j} = \mathbf{j}_{RE} + \mathbf{j}_{ES}. \tag{1.4}$$

Using the co-moving frame which is equivalent to the change of variable $h - Ft \to h$, we obtain from (1.1)–(1.4) the growth equations with a high-order surface relaxation and the ES effect:

$$\partial_t h = (-1)^{m-1} M_m \Delta^m h - \nabla \cdot \left(\frac{F}{2} \frac{\nabla h}{|\nabla h|^2} \right) \qquad \text{for an infinite ES barrier;} \tag{1.5}$$

$$\partial_t h = (-1)^{m-1} M_m \Delta^m h - \nabla \cdot \left(\frac{F S_c \sigma^2 \nabla h}{1+\alpha^2\sigma^2|\nabla h|^2} \right) \qquad \text{for a finite ES barrier.} \tag{1.6}$$

Setting

$$H(X,T) = \eta h(x,t), \quad X = \xi x, \quad T = \zeta t, \tag{1.7}$$

with

$$\xi = \left(\frac{F}{2M_m}\right)^{1/(2m)}, \qquad \eta = 1, \qquad \zeta = M_m \xi^{2m}, \qquad \text{for an infinite ES barrier,}$$

$$\xi = \left(\frac{F S_c \sigma^2}{M_m}\right)^{1/(2m-2)}, \qquad \eta = \alpha\sigma\xi, \qquad \zeta = M_m \xi^{2m}, \qquad \text{for a finite ES barrier,}$$

we obtain from (1.5) and (1.6) the following equations for the rescaled height $H = H(X,T)$ but written using $h = h(x,t)$ instead for convenience:

$$\partial_t h = (-1)^{m-1}\Delta^m h - \nabla\cdot\left(\frac{\nabla h}{|\nabla h|^2}\right) \qquad \text{for an infinite ES barrier;} \tag{1.8}$$

$$\partial_t h = (-1)^{m-1}\Delta^m h - \nabla\cdot\left(\frac{\nabla h}{1+|\nabla h|^2}\right) \qquad \text{for a finite ES barrier.} \tag{1.9}$$

We shall consider these equations in d-dimensional Euclidean space $\mathbb{R}^d$ for some $d \geq 1$ with a periodical boundary condition. Let $Q \subset \mathbb{R}^d$ be the open periodical cell, an open cube in $\mathbb{R}^d$ with its faces parallel to the coordinate planes. Denote by $\overline{Q}$ the closure of Q. If $h = h(x,t)$ is smooth, $\overline{Q}$-periodical in x, and satisfies (1.8) or (1.9), then

$$\frac{d}{dt}\int_Q h(x,t)\,dx = 0, \tag{1.10}$$

i.e., the mass is conserved.

Let us denote for any integer $k \geq 0$ and any function u which has all the derivatives up to order k

$$W_k(u) = \begin{cases} \left|\Delta^{k/2}u\right|^2 & \text{if } k \text{ is even,} \\ \left|\nabla\Delta^{(k-1)/2}u\right|^2 & \text{if } k \text{ is odd,} \end{cases} \tag{1.11}$$

where $\Delta^0 u = u$. We can verify that the equations (1.8) and (1.9) with the $\overline{Q}$-periodical boundary condition are formally the gradient flows of the following effective free-energy functionals, respectively:

$$I(h) = \fint_Q \left[\frac{1}{2}W_m(h) - \log|\nabla h|\right] dx \qquad \text{for an infinite ES barrier;} \tag{1.12}$$

$$J(h) = \fint_Q \left[\frac{1}{2}W_m(h) - \frac{1}{2}\log\left(1+|\nabla h|^2\right)\right] dx \qquad \text{for a finite ES barrier.} \tag{1.13}$$

Here and below,

$$\fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx \tag{1.14}$$

denotes the mean value of a Lebesgue integrable function $u : E \to \mathbb{R}$ with E a d-dimensional Lebesgue measurable set with a finite Lebesgue measure $|E| > 0$.

It is clear by (1.12) or (1.13) that a low-energy profile should have a large slope which is balanced by the relaxation term. The surface slop should thus increase with time during the dynamics governed by (1.8) or (1.9).

Our main results are as follows:

(1) For each of the energy functionals I and J, the infimum is attained in a suitable Sobolev space. Moreover, if $L > 0$ is the linear size of the periodical cell Q, then the minimum energy for I or J is

$$-(m-1)\log L + O(1) \qquad \text{as } L \to \infty;$$

and for any energy minimizer h of I or J,

$$\sqrt{\fint_Q |\nabla^k h|^2 \, dx} = O\left(L^{m-k}\right), \qquad k = 0, \ldots, m, \qquad \text{as } L \to \infty,$$

where ∇^k represents all the derivatives of order k, cf. (2.1). See Theorem 2.1 and Corollary 2.1;

(2) If L is the linear size of the periodical cell Q, then there exists an L/n-minimizer of the energy functional I or J for each integer $n \geq 1$; and such an L/n- minimizer is in fact an equilibrium solution in the case of a finite ES barrier. Here, we define a λ-minimizer for $\lambda > 0$ to be an energy-minimizer among all admissible functions that are $[0, \lambda]^d$-periodical. See Definition 2.1 and Theorem 2.2;

(3) As the system size increases, a new energy functional renormalized from I is the Γ-limit of those renormalized from J. See Theorem 3.1 and Theorem 3.2;

(4) The initial-boundary-value problem of the growth equation (1.9) with a periodical boundary condition is well-posed. See Theorem 4.1;

(5) If $h = h(x, t)$ is a smooth, $\overline{Q}$-periodical solution of (1.9), then

$$\begin{aligned} &w_h(t) \leq C t^{1/2}, \\ &\sqrt{\fint_{t_0}^{t} \fint_Q |\nabla^k h(x, \tau)|^2 \, dx d\tau} \leq C t^{(m-k)/(2m)}, \qquad k = 1, \ldots, m, \\ &I(h(\cdot, t)) \geq -\frac{m-1}{2m} \log t + C, \end{aligned}$$

where $w_h(t)$ is the interface width of h (cf. (5.1)) and C is a generic constant that is independent of h, Q, and t. See Theorem 5.1.

If $h = h(x, t)$ is a solution of (1.6) with the given parameters M_m, F, S_c, σ, and α, then we find that for any $t_0 \geq 0$ and $t > t_0$ large enough

$$w_h(t) \leq \sqrt{\frac{2FS_c}{\alpha^2} t + C},$$

$$\sqrt{\fint_{t_0}^{t} \fint_Q W_k(h(x,\tau))\, dx d\tau} \le \frac{\sqrt{FS_c}}{\alpha M_m^{k/(2m)}} t^{\frac{m-k}{2m}} + C, \qquad k = 1, \ldots, m,$$

where $C > 0$ is a constant independent of h, Q, and t. See Corollary 5.1.

Part (1) and Part (2) above are parallel in part to the related results in [20]. They are obtained by studying the rescaled, singularly perturbed energy functionals; cf. (2.7) and (2.8). Our results here, however, are valid for any integer $m \ge 2$ and for both the functionals I and J. These results are optimal and their proofs are much refined. Note that the term $-\log |\nabla h|$ in the energy functional I defined in (1.12) has to be treated with care. The concept of λ-minimizers, though first introduced here, has been used implicitly in [20, 25] to predict scaling laws for the coarsening dynamics in epitaxial growth. See more discussions in Section 6.

It is important to treat the spacial case of an infinite ES barrier which is studied in [10] to obtain the scaling laws. Our Γ-convergence result in Part (3) indicates that the ES effect is enhanced in a large system. This is because the slope of a film surface can become large—and hence the ES instability can be fully developed—only in a large-system. Consequently, a finite ES barrier can be regarded effectively as an infinite one in a large system. Some of the techniques used in proving the results in Parts (1)–(3) are developed in our recent work [18].

The well-posedness in Part (4) provides a basis for any of the growth scaling laws to be mathematically meaningful. At this point, the well-posedness of the initial-boundary-value problem for the equation (1.8) is not obtained.

Finally, our bounds in Part (5) are only one-sided. Two-sided bounds can never be valid for steady-state solutions, and hence, for all the solutions. Our method for proving the upper bounds in Part (5) is different from that in [13, 14]. In our case, there are two independent quantities: one is the surface height and the other the lateral size of mounds (or, independently, the surface slope). Moreover, the surface slope can increase and the energy can decrease unbounded with respect to the increase of the system size. Our argument is elementary, and is based on observations on the relation between the height profile and its gradients.

The rest of this paper is organized as follows: In Section 2, we study the large-system asymptotics as well as λ-minimizers of the energy functionals (1.12) and (1.13); In Section 3, we show the Γ-convergence of renormalized energies; In Section 4, we prove the well-posedness of the initial-boundary-value problem of Eq. (1.9); In Section 5, we give bounds on the interface width, gradients, and energy for solutions of Eq. (1.9); Finally, in Section 6, we further discuss the result of our analysis.

2 Energy Minimization

In this section, we study the asymptotics of "ground states" and special equilibriums, the λ-minimizers. These variational properties can be used to predict scaling laws; cf. Section 6.

Let us fix a cube $Q \subset \mathbb{R}^d$ as above. Denote by $C^\infty_{per}(\overline{Q})$ the set of all real-valued, $\overline{Q}$-periodic, C^∞-functions on $\mathbb{R}^d$. For any integer $k \geq 1$, let $H^k_{per}(Q)$ be the closure in the usual Sobolev space $H^k(Q) = W^{k,2}(Q)$ of the set of all functions in $C^\infty_{per}(\overline{Q})$ restricted onto $\overline{Q}$ [1, 9]. As usual, we denote $H^0(Q) = L^2(Q)$. By (1.10), we can always assume that the constant mean-value of h over Q is in fact 0. Thus, we introduce

$$\mathring{H}^k_{per}(Q) = \left\{ u \in H^k_{per}(Q) : \int_Q u\, dx = 0 \right\}.$$

It is clear that $\mathring{H}^k_{per}(Q)$ is a closed subspace of $H^k_{per}(Q)$.

For any integer $k \geq 1$ and any $u : Q \to \mathbb{R}$ that has all the weak derivatives of order k, we define $|\nabla^k u|$ by

$$|\nabla^k u|^2 = \sum_{|\beta|=k} |\partial^\beta u|^2, \tag{2.1}$$

where β in the sum is a d-dimensional index. As usual, $|\nabla^0 u|^2 = |u|^2$ for any function u. Note that $\nabla^2 \neq \Delta$ for $d \geq 2$. By the Poincaré inequality, there exist constants $K_1(k, Q) > 0$ and $K_2(k, Q) > 0$ such that

$$K_1(k, Q)\|u\|_{H^k(Q)} \leq \|\nabla^k u\|_{L^2(Q)} \leq K_2(k, Q)\|u\|_{H^k(Q)} \qquad \forall u \in \mathring{H}^k_{per}(Q). \tag{2.2}$$

By integration by parts, we have (cf. Lemma 3.1 in [19])

$$\int_Q |\Delta u|^2 dx = \int_Q |\nabla^2 u|^2 dx \qquad \forall u \in H^2_{per}(Q).$$

Thus, it follows from (1.11) and (2.2) that there exist constants $K_3(k, Q) > 0$ and $K_4(k, Q) > 0$ such that

$$K_3(k, Q)\|u\|_{H^k(Q)} \leq \sqrt{\int_Q W_k(u)\, dx} \leq K_4(k, Q)\|u\|_{H^k(Q)} \qquad \forall u \in \mathring{H}^k_{per}(Q). \tag{2.3}$$

Associated with the integral in (2.3) is the bilinear form $A : H^k_{per}(Q) \times H^k_{per}(Q) \to \mathbb{R}$, defined by

$$A_k(u, v) = \begin{cases} \displaystyle\int_Q \Delta^{k/2} u\, \Delta^{k/2} v\, dx & \text{if } k \text{ is even} \\ \displaystyle\int_Q \nabla \Delta^{(k-1)/2} u \cdot \nabla \Delta^{(k-1)/2} v\, dx & \text{if } k \text{ is odd} \end{cases} \qquad \forall u, v \in H^k_{per}(Q). \tag{2.4}$$

Clearly,

$$A_k(u,u) = \int_Q W_k(u)\,dx \qquad \forall u \in H^k_{per}(Q). \tag{2.5}$$

To study the large-system asymptotics, we need to scale the underlying domain Q in the definition of the functionals I and J (cf. (1.12) and (1.13)) to a fixed domain. Thus, assuming $Q = (0, L)^d$ and setting $\varepsilon = 1/L$, we rescale the energy functionals (1.12) and (1.13) to get

$$I(\hat{h}) = I_\varepsilon(h) \quad \text{and} \quad J(\hat{h}) = J_\varepsilon(h) \quad \text{with} \quad h(x) = \varepsilon \hat{h}(\hat{x}) \quad \text{and} \quad x = \varepsilon \hat{x}, \tag{2.6}$$

where

$$I_\varepsilon(h) = \int_{Q_1} \left[\frac{\varepsilon^{2(m-1)}}{2} W_m(h) - \log|\nabla h| \right] dx, \tag{2.7}$$

$$J_\varepsilon(h) = \int_{Q_1} \left[\frac{\varepsilon^{2(m-1)}}{2} W_m(h) - \frac{1}{2}\log\left(1 + |\nabla h|^2\right) \right] dx, \tag{2.8}$$

and $Q_1 = (0,1)^d$ is the unit cube in $\mathbb{R}^d$.

Throughout the rest of the paper, we fix the integers $d \geq 1$ and $m \geq 2$, and the constant $L > 0$. We also use the notation

$$Q = (0, L)^d, \qquad Q_1 = (0,1)^d, \qquad \mathcal{H}(Q) = \mathring{H}^m_{per}(Q), \qquad \mathcal{H} = \mathring{H}^m_{per}(Q_1). \tag{2.9}$$

The following theorem gives the optimal asymptotics of the minimum energy and magnitude of gradients of energy minimizers for both of the singularly perturbed functionals (2.7) and (2.8):

Theorem 2.1 (1) *For any $\varepsilon > 0$, the infimum of $I_\varepsilon : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ and that of $J_\varepsilon : \mathcal{H} \to \mathbb{R}$ are finite and attained. Moreover, for any $\varepsilon \in (0, 1]$*

$$(m-1)\log\varepsilon + \min_{u \in \mathcal{H}} J_1(u) \leq \min_{h \in \mathcal{H}} J_\varepsilon(h) \leq \min_{h \in \mathcal{H}} I_\varepsilon(h) = (m-1)\log\varepsilon + \min_{u \in \mathcal{H}} I_1(u). \tag{2.10}$$

(2) *There exist constants $C_1 > 0$, $C_2 > 0$, and $\varepsilon_0 \in (0, 1]$, all depending only on d and m, such that for any energy minimizer $h \in \mathcal{H}$ of $I_\varepsilon : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ or $J_\varepsilon : \mathcal{H} \to \mathbb{R}$ and any $\varepsilon \in (0, \varepsilon_0]$,*

$$C_1 \varepsilon^{1-m} \leq \|\nabla^k h\|_{L^2(Q_1)} \leq C_2 \varepsilon^{1-m}, \qquad k = 0, \ldots, m.$$

To prove this theorem, we recall the following result (cf. Lemma 3.1 in [18]) that shows the lower semicontinuity of the logarithmic part of the energy functional (2.7) and the finiteness of such energy of a limiting function:

Lemma 2.1 *Let $E \subset \mathbb{R}^d$ be Lebesgue measurable with $0 < |E| < \infty$. Suppose $g_j \to g$ in $L^1(E)$ and $\left\{\int_E \log|g_j|\,dx\right\}_{j=1}^{\infty}$ is bounded. Then, $\log|g| \in L^1(E)$ and*

$$\liminf_{j\to\infty}\left(-\int_E \log|g_j|\,dx\right) \geq -\int_E \log|g|\,dx. \tag{2.11}$$

Proof of Theorem 2.1. (1) Fix $\varepsilon > 0$. It follows from (2.2), (2.3), and the fact that $m \geq 2$ that there exits a constant $C_3 = C_3(d,m) > 0$ such that

$$\int_{Q_1} |\nabla h|^2 dx \leq C_3^2 \int_{Q_1} W_m(h)\,dx \qquad \forall h \in \mathcal{H}. \tag{2.12}$$

Sine $(1/s)\log(1+s) \to 0$ as $s \to \infty$, there exists $R = R(d,m,\varepsilon) \geq 1$ such that

$$\log(1+s) \leq \varepsilon^{2(m-1)} s/(2C_3^2) \qquad \forall s \geq R.$$

Consequently, we have by (2.12) and (2.3) that

$$\begin{aligned}
I_\varepsilon(h) &\geq J_\varepsilon(h) \\
&= \int_{Q_1} \frac{\varepsilon^{2(m-1)}}{2} W_m(h)\,dx - \left(\int_{\{x\in Q_1:|\nabla h|<R\}} + \int_{\{x\in Q_1:|\nabla h|\geq R\}}\right) \frac{1}{2}\log(1+|\nabla h|^2)\,dx \\
&\geq \frac{\varepsilon^{2(m-1)}}{2}\int_{Q_1} W_m(h)dx - \frac{1}{2}\log(1+R^2) - \frac{\varepsilon^{2(m-1)}}{4C_3^2}\int_{Q_1}|\nabla h|^2 dx \\
&\geq \frac{\varepsilon^{2(m-1)}(K_3(m,Q_1))^2}{4}\|h\|^2_{H^m(Q_1)} - \frac{1}{2}\log(1+R^2) \qquad \forall h \in \mathcal{H}.
\end{aligned} \tag{2.13}$$

Set $\mu_\varepsilon = \inf_{h\in\mathcal{H}} I_\varepsilon(h)$ and $\nu_\varepsilon = \inf_{h\in\mathcal{H}} J_\varepsilon(h)$. Clearly, both μ_ε and ν_ε are finite. Let $\{h_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ be infimizing sequences of $I_\varepsilon : \mathcal{H} \to \mathbb{R}\cup\{\infty\}$ and $J_\varepsilon : \mathcal{H} \to \mathbb{R}$, respectively. It follows from (2.13) that both of these sequences are bounded in $\mathcal{H}$. Thus, up to subsequences, $h_j \rightharpoonup h_\varepsilon$ in $\mathcal{H}$ and $h_j \to h_\varepsilon$ in $H^1(Q_1)$, and $g_j \rightharpoonup g_\varepsilon$ in $\mathcal{H}$ and $g_j \to g_\varepsilon$ in $H^1(Q_1)$, for some $h_\varepsilon \in \mathcal{H}$ and $g_\varepsilon \in \mathcal{H}$, respectively, where the symbol $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively. It is easy to see from (2.3), (2.7), and (2.13) that $\{\int_{Q_1} \log|\nabla h_j|\,dx\}_{j=1}^\infty$ is bounded. Thus, by the fact that $W_m(h)$ is quadratic and convex in h and by Lemma 2.1,

$$\mu_\varepsilon = \liminf_{j\to\infty} I_\varepsilon(h_j) \geq I(h_\varepsilon) \geq \mu_\varepsilon. \tag{2.14}$$

By the fact that $\log(1+s) \leq s$ for all $s \geq 0$ and the Cauchy-Schwarz inequality, we have

$$\left|\int_{Q_1}\left[\log(1+|\nabla g_j|^2) - \log(1+|\nabla g_\varepsilon|^2)\right]dx\right| = \left|\int_{Q_1}\log\left(1+\frac{|\nabla g_j|^2 - |\nabla g_\varepsilon|^2}{1+|\nabla g_\varepsilon|^2}\right)dx\right|$$

$$\leq \int_{Q_1} \left| \frac{|\nabla g_j|^2 - |\nabla g_\varepsilon|^2}{1+|\nabla g_\varepsilon|^2} \right| dx \leq \left(\|\nabla g_j\|_{L^2(Q_1)} + \|\nabla g_\varepsilon\|_{L^2(Q_1)} \right) \|\nabla g_j - \nabla g_\varepsilon\|_{L^2(Q_1)} \to 0$$

as $j \to \infty$. Consequently,

$$\nu_\varepsilon = \liminf_{j\to\infty} J_\varepsilon(g_j) \geq J(g_\varepsilon) \geq \nu_\varepsilon. \tag{2.15}$$

The attainment of the infimum of $I_\varepsilon : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ and that of $J_\varepsilon : \mathcal{H} \to \mathbb{R}$ now follow from (2.14) and (2.15).

Fix $\varepsilon \in (0, 1]$. For each $h \in \mathcal{H}$, let $u = \varepsilon^{m-1} h \in \mathcal{H}$. Then, we have by (2.7) and (2.8) that

$$(m-1)\log\varepsilon + J_1(u) \leq J_\varepsilon(h) \leq I_\varepsilon(h) = (m-1)\log\varepsilon + I_1(u),$$

leading to (2.10).

(2) Let first $h \in \mathcal{H}$ be a minimizer of $I_\varepsilon : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. The function $\xi(s) := I_\varepsilon(h+sh)$, $s \in (-1/2, 1/2)$, is smooth and attains its minimum at $s = 0$. Thus, $\xi'(0) = 0$, i.e.,

$$\int_{Q_1} \varepsilon^{2(m-1)} W_m(h)\, dx = 1.$$

This, together with (2.3), implies that

$$\|h\|_{H^m(Q_1)} \leq C_4 \varepsilon^{1-m}, \tag{2.16}$$

where $C_4 = 1/K_3(m, Q_1) > 0$.

Now, applying Jensen's inequality to the convex function $-log(\cdot)$ and using the upper bound in (2.10) with $C_5 := \min_{u\in\mathcal{H}} I_1(u) \in \mathbb{R}$, we have

$$C_5 + (m-1)\log\varepsilon = I_\varepsilon(h) \geq \int_{Q_1} -\frac{1}{2}\log|\nabla h|^2 dx \geq -\frac{1}{2}\log\left(\int_{Q_1} |\nabla h|^2 dx\right).$$

This, together with an integration by parts, the Cauchy-Schwarz inequality, (2.16) and the fact that $m \geq 2$, implies

$$\begin{aligned} e^{-2C_5}\varepsilon^{2(1-m)} &\leq \int_{Q_1} |\nabla h|^2 dx = \int_{Q_1} (-h)\Delta h\, dx \\ &\leq \left(\int_{Q_1} |h|^2 dx\right)^{1/2} \left(\int_{Q_1} |\Delta h|^2 dx\right)^{1/2} \leq C_4\varepsilon^{1-m} \left(\int_{Q_1} |h|^2 dx\right)^{1/2}. \end{aligned} \tag{2.17}$$

Therefore,

$$\|h\|_{L^2(Q_1)} \geq C_4^{-1} e^{-2C_5} \varepsilon^{1-m}. \tag{2.18}$$

Let now $g \in \mathcal{H}$ be a minimizer of $J_\varepsilon : \mathcal{H} \to \mathbb{R}$. Thus, the first variation of J_ε at g vanishes: $\delta J_\varepsilon(g)(\hat{g}) = 0$ for any $\hat{g} \in \mathcal{H}$. In particular, $\delta J_\varepsilon(g)(g) = 0$. Therefore,

$$\varepsilon^{2(m-1)} \int_{Q_1} W_m(g)\, dx = \int_{Q_1} \frac{|\nabla g|^2}{1 + |\nabla g|^2} \leq 1.$$

This, together with (2.3), leads to

$$\|g\|_{H^m(Q_1)} \leq C_4 \varepsilon^{1-m} \tag{2.19}$$

with the same constant C_4 as in (2.16). Consequently, applying Jensen's inequality to $-log(\cdot)$ and using the upper bound in (2.10) with $C_5 = \min_{u\in\mathcal{H}} I_1(u) \in \mathbb{R}$, we obtain

$$C_5 + (m-1)\log\varepsilon \geq J_\varepsilon(g) \geq \int_{Q_1} -\frac{1}{2}\log(1+|\nabla g|^2)\, dx \geq -\frac{1}{2}\log\left(1 + \int_{Q_1} |\nabla g|^2 dx\right),$$

leading to

$$\int_{Q_1} |\nabla g|^2 dx \geq \frac{1}{2} e^{-2C_5} \varepsilon^{2(1-m)} \qquad \text{if } 0 < \varepsilon \leq \varepsilon_0 := \left(2e^{2C_5}\right)^{\frac{1}{2(1-m)}}.$$

By this and (2.19), and by the argument similar to that in (2.17) and (2.18), we have

$$\|g\|_{L^2(Q_1)} \geq (2C_4)^{-1} e^{-2C_5} \varepsilon^{1-m} \qquad \forall \varepsilon \in (0, \varepsilon_0]. \tag{2.20}$$

Now Part (2) of the theorem follows from (2.16), (2.18)–(2.20), and (2.2), with

$$\begin{aligned} C_1 &= (2C_4)^{-1} e^{-2C_5} \min\left(1, K_1(1,Q_1), \ldots, K_1(m,Q_1)\right) > 0, \\ C_2 &= C_4 \max\left(1, K_2(1,Q_1), \ldots, K_2(m,Q_1)\right) > 0. \end{aligned}$$

Q.E.D.

The following result on the "ground states" for the original functionals $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ and $J : \mathcal{H}(Q) \to \mathbb{R}$ is a direct consequence of Theorem 2.1 and the change of variables (2.6); cf. (2.9) for notation:

Corollary 2.1 (1) *The infimum of the energy functional $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ and that of $J : \mathcal{H}(Q) \to \mathbb{R}$ are finite and attained. Moreover, for any $L \geq 1$,*

$$-(m-1)\log L + \min_{u\in\mathcal{H}} J_1(u) \leq \min_{h\in\mathcal{H}(Q)} J(h) \leq \min_{h\in\mathcal{H}(Q)} I(h) = -(m-1)\log L + \min_{u\in\mathcal{H}} I_1(u).$$

(2) *Let C_1, C_2, and ε_0 be the same as in Theorem 2.1. We have for any energy minimizer $h \in \mathcal{H}(Q)$ of $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ or $J : \mathcal{H}(Q) \to \mathbb{R}$ and any $L \geq 1/\varepsilon_0$ that*

$$C_1 L^{m-k} \leq \left(⨍_Q |\nabla^k h|^2\, dx \right)^{1/2} \leq C_2 L^{m-k}, \qquad k = 0, \cdots, m.$$

We now give the definition of λ-minimizers, and prove their existence for the functionals $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ and $J : \mathcal{H}(Q) \to \mathbb{R}$. See Section 6 for more discussions on the related result. Let $n \geq 1$ be an integer, $\lambda = L/n$, and $Q_\lambda = (0, \lambda)^d$. Define

$$\mathcal{H}_\lambda(Q) = \big\{ h \in \mathcal{H}(Q) : \text{ there exist } \phi_k \in C^\infty_{per}(\overline{Q}_\lambda),\ k = 1, \dots, \\ \text{such that } \phi_k \to h \text{ in } H^m(Q) \big\}\,.$$

Definition 2.1 *A function $h \in \mathcal{H}(Q)$ is a λ-minimizer of $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ (or $J : \mathcal{H}(Q) \to \mathbb{R}$), if $h \in \mathcal{H}_\lambda(Q)$ and*

$$I(h) \leq I(g) \quad (\text{or } J(h) \leq J(g)) \qquad \forall g \in \mathcal{H}_\lambda(Q).$$

Theorem 2.2 *For any integer $n \geq 1$, there exist $h_n \in \mathcal{H}(Q)$ and $g_n \in \mathcal{H}(Q)$ that are L/n-minimizers of $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$ and $J : \mathcal{H}(Q) \to \mathbb{R}$, respectively. Moreover, $g_n \in C^\infty(Q)$ and g_n is an equilibrium solution of Eq. (1.9), i.e.,*

$$(-1)^m \Delta^m g_n + \nabla \cdot \left(\frac{\nabla g_n}{1 + |\nabla g_n|^2} \right) = 0 \qquad \text{in } Q. \tag{2.21}$$

Proof. Fix an integer $n \geq 1$ and let $\lambda = L/n$. Define $\hat{J}_n : \mathcal{H}(Q) \to \mathbb{R}$ by

$$\hat{J}_n(\hat{g}) = \fint_Q \left[\frac{n^{2(m-1)}}{2} W_m(\hat{g}) - \frac{1}{2} \log\left(1 + |\nabla \hat{g}|^2\right) \right] dx \qquad \forall\, \hat{g} \in \mathcal{H}(Q).$$

By the proof of Theorem 2.1, we see that there exists a global minimizer $\hat{g}_n \in \mathcal{H}(Q)$ of $\hat{J}_n : \mathcal{H}(Q) \to \mathbb{R}$. Now, by the definition of the space $\mathcal{H}(Q)$, we can extend $\hat{g}_n$ to be almost everywhere $\overline{Q}$-periodical on $\mathbb{R}^d$, in the sense that $\hat{g}_n(x + Le_j) = \hat{g}_n(x)$ for a. e. $x \in \mathbb{R}^d$ and for any unit coordinate vector e_j $(j = 1, \dots, d)$. Define $g_n(x) = (1/n)\hat{g}_n(nx)$ for $x \in \mathbb{R}^d$. It is easy to see that $g_n \in \mathcal{H}_\lambda(Q)$. Moreover, for any $g \in \mathcal{H}_\lambda(Q)$, we can extend g to be almost everywhere $\overline{Q}_\lambda$-periodical on $\mathbb{R}^d$ and define $\hat{g}$ by the relation $g(x) = (1/n)\hat{g}(nx)$ for $x \in \mathbb{R}^d$. Clearly, $\hat{g} \in \mathcal{H}(Q)$, when $\hat{g}$ is restricted onto Q. Further, we can verify that

$$J(g) = \hat{J}_n(\hat{g}) \geq \hat{J}_n(\hat{g}_n) = J(g_n).$$

Thus, g_n is an L/n-minimizer of $J : \mathcal{H}(Q) \to \mathbb{R}$. A similar argument shows the existence of an L/n-minimizer for $I : \mathcal{H}(Q) \to \mathbb{R} \cup \{\infty\}$.

Note that $\hat{g}_n \in \mathcal{H}(Q) \subset H^m_{per}(Q)$ is in fact a global minimizer of $\hat{J}_n : H^m_{per}(Q) \to \mathbb{R}$, since $\hat{J}(h) = \hat{J}(h - \bar{h})$ and $h - \bar{h} \in \mathcal{H}(Q)$ for any $h \in H^m_{per}(Q)$, where $\bar{h}$ is the mean value of h over Q, cf. (1.14). Thus, by simple calculations, we see that $\hat{g}_n$ is a weak solution of the following equation:

$$n^{2(m-1)}(-1)^m \Delta^m \hat{g}_n + \nabla \cdot \left(\frac{\nabla \hat{g}_n}{1 + |\nabla \hat{g}_n|^2} \right) = 0 \qquad \text{in } Q. \tag{2.22}$$

Equivalently, g_n is a weak solution of (2.21). Notice that $\Delta^k g_n = \Delta(\Delta^{k-1} g_n)$ for $1 \leq k \leq m$. Thus, by the regularity theory of elliptic problems and the standard boot-strapping argument, we see that g_n is smooth and satisfies Eq. (2.21) pointwise. **Q.E.D.**

3 Γ-Convergence of Renormalized Energies

In this section, we present our mathematical results using the notion of Γ-convergence. These results indicate that, for a large system, the energy functional for a finite ES barrier is close to that for an infinite ES barrier. We in fact prove a result that is stronger than the usual Γ-convergence: any sequence of energy minimizers has a subsequence that converges *strongly* to a minimizer of the Γ-limit functional; cf. Theorem 3.2.

We define for each $\varepsilon > 0$ the renormalized energy functionals

$$\tilde{I}_\varepsilon(u) = I_\varepsilon(\varepsilon^{1-m}u) - (m-1)\log\varepsilon = \int_{Q_1}\left[\frac{1}{2}W_m(u) - \log|\nabla u|\right]dx, \tag{3.1}$$

$$\tilde{J}_\varepsilon(u) = J_\varepsilon(\varepsilon^{1-m}u) - (m-1)\log\varepsilon = \int_{Q_1}\left[\frac{1}{2}W_m(u) - \frac{1}{2}\log\left(\varepsilon^{2(m-1)} + |\nabla u|^2\right)\right]dx. \tag{3.2}$$

Note that $\tilde{I}_\varepsilon = I_1$ is independent of ε. For convenience, we shall write $\tilde{I} = \tilde{I}_\varepsilon$.

Theorem 3.1 *The energy functionals $\tilde{J}_\varepsilon : \mathcal{H} \to \mathbb{R}$ $(0 < \varepsilon \le 1)$ Γ-converge to $\tilde{I} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ as $\varepsilon \to 0$ with respect to the weak convergence in $\mathcal{H}$.*

The precise definition of the Γ-convergence in the theorem is as follows [5]: For any decreasing sequence $\{\varepsilon_j\}_{j=1}^\infty$ in $(0, 1]$ such that $\lim_{j\to\infty}\varepsilon_j = 0$, the following hold true:

(1) If $u_j \rightharpoonup u$ in $\mathcal{H}$, then

$$\liminf_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) \ge \tilde{I}(u); \tag{3.3}$$

(2) For any $v \in \mathcal{H}$, there exist $v_j \in \mathcal{H}$ $(j = 1, \ldots)$ such that $v_j \rightharpoonup v$ in $\mathcal{H}$ and

$$\lim_{j\to\infty} \tilde{J}_{\varepsilon_j}(v_j) = \tilde{I}(v). \tag{3.4}$$

Proof. (1) Let $u_j \rightharpoonup u$ in $\mathcal{H}$. We may assume that $\liminf_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) < \infty$, for otherwise (3.3) holds trivially. We may further assume, up to a subsequence, that

$$\liminf_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) = \lim_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) < \infty. \tag{3.5}$$

Since $0 < \varepsilon_j \le 1$, $\tilde{J}_{\varepsilon_j}(u_j) \ge J_1(u_j)$ for all $j \ge 1$. Thus, by (2.13) with $\varepsilon = 1$ and (3.5), the sequence $\{\tilde{J}_{\varepsilon_j}(u_j)\}_{j=1}^\infty$ is bounded and $\{u_j\}_{j=1}^\infty$ is bounded in $\mathcal{H}$. Consequently, by (2.3),

$$\left\{\int_{Q_1} \log\sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2}\,dx\right\}_{j=1}^\infty = \left\{\int_{Q_1} \frac{1}{2}W_m(u_j)\,dx - \tilde{J}_{\varepsilon_j}(u_j)\right\}_{j=1}^\infty \tag{3.6}$$

is bounded. Moreover, since $\{u_j\}_{j=1}^\infty$ is bounded in $\mathcal{H}$ and $m \geq 2$, up to a further subsequence, $u_j \to u$ in $H^1(Q_1)$. Thus, since

$$\left|\sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2} - |\nabla u|\right|^2 = \varepsilon_j^{2(m-1)} + |\nabla u_j|^2 + |\nabla u|^2 - 2|\nabla u|\sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2}$$
$$\leq \varepsilon_j^{2(m-1)} + |\nabla u_j - \nabla u|^2 \qquad \forall j \geq 1,$$

we have $\sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2} \to |\nabla u|$ in $L^2(Q_1)$ as $j \to \infty$. Therefore, it follows from Lemma 2.1 that $\log|\nabla u| \in L^1(Q_1)$ and

$$\liminf_{j\to\infty} \left(-\int_{Q_1} \log\sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2} dx \right) \geq -\int_{Q_1} \log|\nabla u|\, dx. \tag{3.7}$$

Since $W_m(\cdot)$ is quadratic and convex, we also have by $u_j \rightharpoonup u$ in $\mathcal{H}$ that

$$\liminf_{j\to\infty} \int_{Q_1} W_m(u_j)\, dx \geq \int_{Q_1} W_m(u)\, dx. \tag{3.8}$$

Now, (3.3) follows from (3.5), (3.7), and (3.8).

(2) Let $v \in \mathcal{H}$ and $v_j = v$ for all integers $j \geq 1$. It follows from Lemma 2.1 that

$$\lim_{j\to\infty} \int_{Q_1} \log\sqrt{\varepsilon_j^{2(m-1)} + |\nabla v|^2}\, dx = \int_{Q_1} \log|\nabla v|\, dx.$$

This implies (3.4). **Q.E.D.**

Corollary 3.1 *We have*

$$\lim_{\varepsilon\to 0} \min_{u\in\mathcal{H}} \tilde{J}_\varepsilon(u) = \min_{u\in\mathcal{H}} \tilde{I}(u). \tag{3.9}$$

Proof. Let $\{\varepsilon_j\}_{j=1}^\infty$ be any decreasing sequence in $(0,1]$ with $\lim_{j\to\infty} \varepsilon_j = 0$. For each $j \geq 1$, let $u_j \in \mathcal{H}$ be a minimizer of $\tilde{J}_{\varepsilon_j} : \mathcal{H} \to \mathbb{R}$. The existence of such a minimizer follows from Theorem 2.1. By (2.8) and (3.2), $\varepsilon_j^{1-m} u_j$ is a minimizer of $J_{\varepsilon_j} : \mathcal{H} \to \mathbb{R}$ for each $j \geq 1$. Thus, by Part (2) of Theorem 2.1, $\{u_j\}_{j=1}^\infty$ is bounded in $\mathcal{H}$. Hence, it has a subsequence, not relabeled, such that $u_j \rightharpoonup u$ in $\mathcal{H}$ for some $u \in \mathcal{H}$. Now, by (3.3), we obtain that

$$\liminf_{j\to\infty} \min_{w\in\mathcal{H}} \tilde{J}_{\varepsilon_j}(w) = \liminf_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) \geq \tilde{I}(u) \geq \min_{w\in\mathcal{H}} \tilde{I}(w). \tag{3.10}$$

Let $v \in \mathcal{H}$ be a minimizer of $\tilde{I} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. By Theorem 3.1, there exist $v_j \in \mathcal{H}$ $(j = 1, \ldots)$ that satisfy (3.4). Thus,

$$\limsup_{j\to\infty} \min_{w\in\mathcal{H}} \tilde{J}_{\varepsilon_j}(w) \leq \limsup_{j\to\infty} \tilde{J}_{\varepsilon_j}(v_j) = \tilde{I}(v) = \min_{w\in\mathcal{H}} \tilde{I}(w). \tag{3.11}$$

Now, (3.9) follows from (3.10), (3.11), and the arbitrariness of $\{\varepsilon_j\}_{j=1}^\infty$. **Q.E.D.**

Theorem 3.2 *Let* $\{\varepsilon_j\}_{j=1}^\infty$ *be a decreasing sequence in* $(0, 1]$ *such that* $\lim_{j\to\infty} \varepsilon_j = 0$. *For each integer* $j \geq 1$, *let* $u_j \in \mathcal{H}$ *be a minimizer of* $\tilde{J}_{\varepsilon_j} : \mathcal{H} \to \mathbb{R}$. *Then, there is a subsequence of* $\{u_j\}_{j=1}^\infty$, *not relabeled, and a minimizer* $u \in \mathcal{H}$ *of* $\tilde{I} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ *such that* $u_j \to u$ (strong convergence) *in* $\mathcal{H}$.

Proof. It follows from (2.8) and (3.2) that $\varepsilon_j^{1-m} u_j$ is a minimizer of $J_{\varepsilon_j} : \mathcal{H} \to \mathbb{R}$ for each integer $j \geq 1$. Thus, by Part (2) of Theorem 2.1, $\{u_j\}_{j=1}^\infty$ is bounded in $\mathcal{H}$. Hence, it has a subsequence, not relabeled, such that $u_j \rightharpoonup u$ in $\mathcal{H}$ and $u_j \to u$ in $H^1(Q_1)$ for some $u \in \mathcal{H}$.

Since $J_1(v) \leq \tilde{J}_{\varepsilon_j}(v) \leq I_1(v)$ for all $v \in \mathcal{H}$, and since both $\min_{v\in\mathcal{H}} J_1(v)$ and $\min_{v\in\mathcal{H}} I_1(v)$ are finite by Theorem 2.1, the sequence $\{\tilde{J}_{\varepsilon_j}(u_j)\}_{j=1}^\infty = \{\min_{v\in\mathcal{H}} \tilde{J}_{\varepsilon_j}(v)\}_{j=1}^\infty$ is bounded. Therefore, the sequence in (3.6) is bounded. By the same argument as in the proof of Theorem 3.1, we have that $\log |\nabla u| \in L^1(Q_1)$ and that (3.7) and (3.8) hold true. Consequently, by Corollary 3.1,

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \min_{v\in\mathcal{H}} \tilde{J}_{\varepsilon_j}(v) - \min_{v\in\mathcal{H}} \tilde{I}(v) \\
&\geq \liminf_{j\to\infty} \tilde{J}_{\varepsilon_j}(u_j) - \tilde{I}(u) \\
&\geq \left[\liminf_{j\to\infty} \int_{Q_1} \frac{1}{2} W_m(u_j)\, dx - \int_{Q_1} \frac{1}{2} W_m(u)\, dx \right] \\
&\quad + \left[\liminf_{j\to\infty} \left(- \int_{Q_1} \log \sqrt{\varepsilon_j^{2(m-1)} + |\nabla u_j|^2}\, dx \right) - \left(- \int_{Q_1} \log |\nabla u|\, dx \right) \right] \\
&\geq 0.
\end{aligned} \tag{3.12}
$$

This implies that $\tilde{I}(u) = \min_{v\in\mathcal{H}} \tilde{I}(v)$, i.e., $u \in \mathcal{H}$ is a minimizer of $\tilde{I} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$.

By (3.7), (3.8), and (3.12), we have

$$
\liminf_{j\to\infty} \int_{Q_1} W_m(u_j)\, dx = \int_{Q_1} W_m(u)\, dx.
$$

Therefore, up to a further subsequence of $\{u_j\}_{j=1}^\infty$, still not relabeled, we have

$$
\lim_{j\to\infty} \int_{Q_1} W_m(u_j)\, dx = \int_{Q_1} W_m(u)\, dx. \tag{3.13}
$$

Note by (1.11) and (2.4) that

$$
\int_{Q_1} W_m(u_j - u)\, dx = \int_{Q_1} W_m(u_j)\, dx + \int_{Q_1} W_m(u)\, dx - 2A_m(u_j, u), \tag{3.14}
$$

where $A_m(\cdot,\cdot)$ is the same as in (2.4) but with $k = m$ and Q replaced by Q_1. Thus, since $u_j \rightharpoonup u$ in $\mathcal{H}$, we have by (3.13), (3.14), the definition of $A_m(\cdot,\cdot)$, and (2.5) that

$$\lim_{j\to\infty} \int_{Q_1} W_m(u_j - u)\, dx = 0.$$

This and (2.3) imply that $u_j \to u$ in $\mathcal{H}$. **Q.E.D.**

4 Well-Posedness

We consider the initial-boundary-value problem of the d-dimensional "growth" equation (1.9) for $h : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ that is $\overline{Q}$-periodical:

$$\partial_t h = (-1)^{m-1}\Delta^m h - \nabla \cdot \left(\frac{\nabla h}{1 + |\nabla h|^2} \right) \qquad \text{in } Q \times (0,T], \tag{4.1}$$

$$h(\cdot,t) \text{ is } \overline{Q}\text{-periodic for all } t \in [0,T], \tag{4.2}$$

$$h(x,0) = h_0(x) \qquad \forall x \in Q, \tag{4.3}$$

where $Q = (0,L)^d$ as before, $T > 0$, and $h_0 \colon Q \to \mathbb{R}$ is a given function. We denote by $H^{-m}_{per}(Q)$ the dual space of $H^m_{per}(Q)$.

Clearly, the definition of a weak solution and Theorem 4.1 below can be generalized to the equation (1.6) with the cube Q replaced by any d-dimensional parallelepiped with its faces parallel to the coordinate planes.

Definition 4.1 *A function $h : Q \times [0,T] \to \mathbb{R}$ is a* weak solution *of the initial-boundary-value problem* (4.1)–(4.3)*, if the following hold true:*

(1) $h \in L^2(0,T; H^m_{per}(Q))$ *and* $\partial_t h \in L^2(0,T; H^{-m}_{per}(Q))$*;*

(2) *for any* $\phi \in H^m_{per}(Q)$*,*

$$\langle \phi, \partial_t h\rangle + A_m(\phi, h) - \langle \nabla\phi, \frac{\nabla h}{1 + |\nabla h|^2}\rangle = 0 \qquad a.e.\ t \in (0,T), \tag{4.4}$$

where, without confusion, $\langle\cdot,\cdot\rangle$ *denotes the value of a linear functional at a function or the inner product of* $L^2(Q)$*, and* $A_m : H^m_{per}(Q) \times H^m_{per}(Q) \to \mathbb{R}$ *is defined in* (2.4)*;*

(3) $h(x,0) = h_0(x)$ *for a.e.* $x \in Q$*.*

The following is the main result in this section:

Theorem 4.1 *Let $h_0 \in H^m_{per}(Q)$. Then, the initial-boundary-value problem* (4.1)–(4.3) *has a unique weak solution $h : \Omega \times [0, T] \to \mathbb{R}$. Moreover, if $g : \Omega \times [0, T] \to \mathbb{R}$ is the weak solution to* (4.1)–(4.3) *with h_0 replaced by $g_0 \in H^m_{per}(Q)$, then*

$$\|g - h\|_{L^\infty(0,T;L^2(Q))} + \|g - h\|_{L^2(0,T;H^m(Q))} \le C\|g_0 - h_0\|_{H^m(Q)}, \tag{4.5}$$

where the constant $C = C(m, d, Q, T) > 0$ is independent of g_0 and h_0.

The proof of this theorem is similar to that in [19]. To be self-complete, we give here a shortened proof. We first need some preparations. Denote for each integer $N \ge 1$

$$H_N = \text{span}\left\{1, \cos\left(\frac{2\pi k \cdot x}{L}\right), \sin\left(\frac{2\pi k \cdot x}{L}\right) : 0 < |k| \le N\right\},$$

where $k = (k_1, \cdots, k_d)$, all $k_j \ge 0$ $(j = 1, \ldots, d)$ are integers, and $|k| = \sum_{j=1}^d k_j$. Notice that $H_N \subset C^\infty_{per}(\overline{Q})$. Denote also by $\mathcal{P}_N : L^2(Q) \to H_N$ the $L^2(Q)$-projection onto H_N, which is defined for any $u \in L^2(Q)$ by $\mathcal{P}_N u \in H_N$ and

$$\langle \mathcal{P}_N u - u, \phi \rangle = 0 \qquad \forall \phi \in H_N.$$

We have for any integer $k \ge 0$ that (cf. Lemma 3.2 of [19])

$$\|\mathcal{P}_N u\|_{H^k(Q)} \le \|u\|_{H^k(Q)} \qquad \forall u \in H^k_{per}(Q),\ \forall N \ge 1, \tag{4.6}$$

$$\lim_{N\to\infty} \|\mathcal{P}_N u - u\|_{H^k(Q)} = 0 \qquad \forall u \in H^k_{per}(Q). \tag{4.7}$$

In what follows, we denote by C a generic, positive constant that, unless otherwise stated, can depend on m, d, Q, T, and h_0, but not on N.

Lemma 4.1 *Let $h_0 \in H^m_{per}(Q)$. For each integer $N \ge 1$, there exists a unique $h_N : Q \times [0, T] \to \mathbb{R}$ such that*

(1) *$h_N \in C^\infty(\overline{Q} \times [0, T])$ and $h_N(\cdot, t) \in H_N$ for any $t \in [0, T]$;*

(2) *for any $\phi \in H_N$ and any $t \in (0, T]$,*

$$\langle \phi, \partial_t h_N \rangle + A_m(\phi, h_N) - \langle \nabla\phi, \frac{\nabla h_N}{1 + |\nabla h_N|^2} \rangle = 0; \tag{4.8}$$

(3) $h_N(\cdot, 0) = \mathcal{P}_N h_0$;

(4) $\|\partial_t h_N\|_{L^2(0,T;L^2(Q))} + \|h_N\|_{L^\infty(0,T;H^m(Q))} \le C.$

Proof. Let $\{\phi_j\}_{j=1}^r$ be an orthonormal basis of H_N with respect to the inner product in $L^2(Q)$, where $r = \dim H_N$. Let $h_N(x,t) = \sum_{j=1}^r a_j(t)\phi_j(x)$ with all $a_j = a_j(t)$ to be determined. Set $\phi = \phi_i$ in (4.8) and use the orthogonality of $\{\phi_j\}_{j=1}^r$ to obtain

$$a_i'(t) = f_i(a_1(t), \dots, a_r(t)), \qquad i = 1, \dots, r, \tag{4.9}$$

where all $f_i \colon \mathbb{R}^r \to \mathbb{R}$ $(1 \le i \le r)$ are smooth and locally Lipschitz. Set

$$a_i(0) = \langle h_0, \phi_i \rangle, \qquad i = 1, \dots, r, \tag{4.10}$$

which is equivalent to Part (3). It follows from the theory for initial-value problems of ordinary differential equations that there exists $T_N > 0$ such that the initial-value problem, (4.9) and (4.10), has a unique smooth solution $(a_1(t), \dots, a_r(t))$ for $t \in [0, T_N]$.

Setting $\phi = h_N(\cdot, t) \in H_N$ in (4.8) and integrating against t, we get from (2.5) and (4.6) that

$$\frac{1}{2}\|h_N(\cdot,t)\|^2 + \int_0^t \int_Q W_m(h_N(x,\tau))dxd\tau \le |Q|T_N + \frac{1}{2}\|h_0\|^2 \qquad \forall t \in [0, T_N]. \tag{4.11}$$

Here and below, we denote by $\|\cdot\|$ the $L^2(Q)$-norm. By the orthogonality of $\{\phi_j\}_{j=1}^r$, we thus obtain that

$$\sum_{j=1}^r [a_j(t)]^2 = \|h_N(\cdot,t)\|^2 \le 2|Q|T_N + \|h_0\|^2 \qquad \forall t \in [0, T_N].$$

The solution $(a_1(t), \dots, a_r(t))$ of the initial-value problem, (4.9) and (4.10), is thus bounded on $[0, T_N]$, and hence can be uniquely extended to a smooth solution over $[0, \infty)$. Parts (1)–(3) are proved.

By (2.3) and the Poincaré inequality, we have

$$\|u\|_{H^m(Q)}^2 \le C\left(\|u\|^2 + \int_Q W_m(u)\,dx\right) \qquad \forall u \in H^m_{per}(Q). \tag{4.12}$$

Thus, replacing T_N by T in (4.11), we obtain that

$$\|h_N\|_{L^\infty(0,T;L^2(Q))} + \|h_N\|_{L^2(0,T;H^m(Q))} \le C. \tag{4.13}$$

Set now $\phi = \partial_t h_N(\cdot,t)$ in (4.8) to get for any $t \in [0,T]$ that

$$\|\partial_t h_N(\cdot,t)\|^2 + |Q|\frac{d}{dt}J(h_N(\cdot,t)) = 0, \tag{4.14}$$

where $J(\cdot)$ is defined in (1.13). Note by Part (3), (2.3), and (4.6) that

$$J(h_N(\cdot,0)) \le C\|\mathcal{P}_N h_0\|_{H^m(Q)} \le C\|h_0\|_{H^m(Q)}. \tag{4.15}$$

Consequently, integrating against t in (4.14), noting that $\ln(1+s^2) \le 2s$ for all $s \ge 0$, using (2.3), (4.6), and (4.15), and applying Young's inequality, we obtain

$$
\begin{aligned}
&\int_0^t \|\partial_t h_N(\cdot,\tau)\|^2 d\tau + \|h_N(\cdot,t)\|^2_{H^m(Q)} \\
&\quad \le C J_N(h_N(\cdot,0)) + C\int_Q \log(1+|\nabla h_N(x,t)|^2 dx \\
&\quad \le C + C\int_Q |\nabla h_N(x,t)|\, dx \\
&\quad \le C + \frac{1}{2}\int_Q |\nabla h_N(x,t)|^2 dx \qquad \forall t \in [0,T].
\end{aligned}
$$

This and (4.13) lead to Part (4). **Q.E.D.**

Proof of Theorem 4.1. Let $h_N \in H_N$ be defined as in Lemma 4.1. Then, there exists a subsequence of $\{h_N\}_{N=1}^\infty$, not relabeled, and $h \in L^\infty(0,T;H^m_{per}(Q))$ with $\partial_t h \in L^2(0,T;L^2(Q))$ such that

$$h_N \overset{*}{\rightharpoonup} h \qquad \text{in } L^\infty(0,T;H^m(Q)), \tag{4.16}$$

$$\partial_t h_N \rightharpoonup \partial_t h \qquad \text{in } L^2(0,T;L^2(Q)), \tag{4.17}$$

$$h_N \to h \qquad \text{in } L^2(0,T;H^1(Q)), \tag{4.18}$$

where the strong convergence (4.18) follows from the combination of the weak convergence $h_N \rightharpoonup h$ in $L^2(0,T;H^2(Q))$ which results from the weak-$\star$ convergence (4.16) and the fact that $m \ge 2$, the weak convergence (4.17), and a usual compactness result (cf. [34], Theorem 2.1, Chapter III). Clearly, Part (1) of Definition 4.1 is satisfied.

Let $\psi \in H^m_{per}(Q)$ and $\eta \in C[0,T]$. For each $N \ge 1$, setting $\phi = \mathcal{P}_N\psi$ in (4.8), multiplying both sides of the resulting identity by $\eta(t)$, and integrating against t, we obtain that

$$\int_0^T \eta(t)\left[\langle \mathcal{P}_N\psi, \partial_t h_N\rangle + A_m(\mathcal{P}_N\psi, h_N) - \langle \mathcal{P}_N\psi, \frac{\nabla h_N}{1+|\nabla h_N|^2}\rangle\right] dt = 0. \tag{4.19}$$

Sending $N \to \infty$, we get by (4.7), (4.16), and (4.18) that

$$\int_0^T \eta(t)\left[\langle \psi, \partial_t h\rangle + A_m(\psi, h) - \langle \psi, \frac{\nabla h}{1+|\nabla h|^2}\rangle\right] dt = 0. \tag{4.20}$$

Since $\eta \in C[0,T]$ is arbitrary, this implies (4.4) with ϕ replaced by ψ. Part (2) of Definition 4.1 is satisfied.

It follows from a standard argument (cf. [9], Theorem 2, Section 5.9) that, after a possible modification of h on a set of measure zero, we have $h \in C([0,T]; L^2(\Omega))$. Moreover, $h(t) = h(s) + \int_s^t h'(\tau)\, d\tau$ for any $s, t \in [0,T]$, where $h(t) = h(\cdot, t) \in L^2(\Omega)$ and $h'(t) = \partial_t h(\cdot, t)$. Replacing $\eta(t)$ in (4.19) and (4.20) by $\eta_T(t) = -t/T + 1$, integrating by parts against t for the first terms in (4.19) and (4.20), and repeating the argument for the passage from (4.19) to (4.20), we get

$$
\begin{aligned}
\langle \psi, h_0 \rangle &= \lim_{N\to\infty} \langle \mathcal{P}_N \psi, h_0 \rangle \\
&= \lim_{N\to\infty} \langle \mathcal{P}_N \psi, h_N(\cdot, 0) \rangle \\
&= \lim_{N\to\infty} \int_0^T \left\{ \frac{1}{T} \langle \mathcal{P}_N \psi, h_N(\cdot, t) \rangle \right. \\
&\qquad\qquad \left. + \eta_T(t) \left[A_m(\mathcal{P}_N \psi, h_N(\cdot, t)) - \langle \mathcal{P}_N \psi, \frac{\nabla h_N(\cdot, t)}{1 + |\nabla h_N(\cdot, t)|^2} \rangle \right] \right\} dt \\
&= \int_0^T \left\{ \frac{1}{T} \langle \psi, h(\cdot, t) \rangle + \eta_T(t) \left[A_m(\psi, h(\cdot, t)) - \langle \psi, \frac{\nabla h(\cdot, t)}{1 + |\nabla h(\cdot, t)|^2} \rangle \right] \right\} dt \\
&= \langle \psi, h(\cdot, 0) \rangle \qquad \forall \psi \in H^m_{per}(Q).
\end{aligned}
$$

Part (3) in Definition 4.1 is satisfied. Thus, h is a weak solution.

Let now $f = g - h$. Since g and h are two weak solutions, we have for any $\psi \in H^m_{per}(Q)$ that

$$
\langle \psi, \partial_t f \rangle + A_m(\psi, f) - \langle \nabla \psi, \frac{\nabla h}{1 + |\nabla h|^2} - \frac{\nabla g}{1 + |\nabla g|^2} \rangle = 0 \qquad \text{for a.e. } t \in (0,T).
$$

Since $f \in L^2(0,T; H^m_{per}(Q))$ and $\partial_t f \in L^2(0,T; H^{-m}_{per}(Q))$, the mapping $t \mapsto \|f(\cdot, t)\|^2$ is absolutely continuous and $\frac{d}{dt}\langle f, f \rangle = 2\langle f, \partial_t f \rangle$; cf. [9], Theorem 3, Section 5.9, with $H^1_0(U)$ and $H^{-1}(U)$ replaced by $H^m_{per}(Q)$ and $H^{-m}_{per}(Q)$, respectively. Setting $\psi = f(\cdot, t)$, we have

$$
\frac{1}{2}\frac{d}{dt}\|f\|^2 + A_m(f, f) = \|\nabla f\|^2 - \langle \nabla h - \nabla g, \frac{|\nabla h|^2}{1 + |\nabla h|^2} \nabla h - \frac{|\nabla g|^2}{1 + |\nabla g|^2} \nabla g \rangle \tag{4.21}
$$

for a.e. $t \in (0,T)$. It is easy to verify for any vectors $a, b \in \mathbb{R}^d$ that (cf. [19])

$$
2(a - b) \cdot \left(\frac{|a|^2 a}{1 + |a|^2} - \frac{|b|^2 b}{1 + |b|^2} \right) = \frac{\left(|a|^2 - |b|^2\right)^2}{(1 + |a|^2)\,(1 + |b|^2)} + |a - b|^2 \left(\frac{|a|^2}{1 + |a|^2} + \frac{|b|^2}{1 + |b|^2} \right).
$$

Setting $a = \nabla g$ and $b = \nabla h$ in (4.21), we then deduce by an integration by parts, the Cauchy-Schwarz inequality, and Young's inequality that

$$
\frac{1}{2}\frac{d}{dt}\|f\|^2 + A_m(f, f) \le \|\nabla f\|^2 = -\int_Q f \Delta f \, dx
$$

$$\leq \|f\| \, \|f\|_{H^m(Q)} \leq \frac{2}{K_3^2}\|f\|^2 + \frac{K_3^2}{2}\|f\|^2_{H^m(Q)},$$

where $K_3 = K_3(m, Q)$ is the same as in (2.3). This, together with (2.5), (2.3), and the Gronwall inequality, leads to (4.5). **Q.E.D.**

5 Bounds for Scaling Laws

For any $h : [0, T] \to L^2(Q)$, we define its interface width for any $t \in [0, T]$ to be

$$w_h(t) = \sqrt{⨍_Q |h(x,t) - \bar{h}(t)|^2 dx} \qquad \text{with } \bar{h}(t) = ⨍_Q h(x,t)\, dx. \tag{5.1}$$

The interface width is readily measurable in laboratory. It describes the fluctuation of surface height. Often it obeys a scaling law $w_h(t) \sim t^\beta$ for some constant $\beta > 0$ called the *growth exponent* [4, 24]. Another quantity that is also of experimentally interest is the characteristic lateral size of mounds formed during the growth of crystalline surfaces. This length increases with time and the system thus coarsens. Intuitively, the lateral size is determined by the surface slope and interface width.

It is important to understand these scaling laws, since they are distinguished by microscopic properties of an underlying growth environment. While proving a strict scaling seems to be impossible (see the discussion in Section 1), in this section, we provide one-sided bounds of some of the scaling laws for the coarsening dynamics predicted by the underlying models. More discussions on our results are provided in Section 6.

Theorem 5.1 *Let $h : [0, \infty) \to H^m_{per}(Q)$ be a weak solution of the initial-boundary-value problem* (4.1)–(4.3) *on* $[0, T]$ *for any* $T > 0$. *Let* $t_0 \geq 0$. *We have*

$$w_h(t) \leq \sqrt{2(t - t_0) + [w_h(t_0)]^2} \qquad \forall\, t \geq t_0, \tag{5.2}$$

$$\sqrt{⨍_{t_0}^t ⨍_Q W_k(h(x,\tau))\, dx d\tau} \leq \left(1 + \frac{[w_h(t_0)]^2}{2(t-t_0)}\right)^{\frac{k}{2m}} \left(t - t_0 + [w_h(t_0)]^2\right)^{\frac{m-k}{2m}} \qquad \forall\, t > t_0,\ k = 1, \ldots, m, \tag{5.3}$$

$$⨍_{t_0}^t E(h(\tau)) d\tau \geq -\frac{1}{2}\log\left(1 + 2^{(m-2)/m} 3^{1/m} (t - t_0)^{(m-1)/m}\right) \qquad \forall\, t > t_0 + [w_h(t_0)]^2. \tag{5.4}$$

To prove Theorem 5.1, we need the following:

Lemma 5.1 *Let $n \geq 1$ be an integer and $A_0, A_1, \dots, A_n$ be $n+1$ positive real numbers. Assume*

$$A_k^2 \leq A_{k-p} A_{k+p} \quad \text{for } p = 0, \dots, \min(k, n-k) \text{ and } k = 0, \dots, n. \tag{5.5}$$

Then

$$A_k^n \leq A_0^{n-k} A_n^k \quad \text{for } k = 0, \dots, n. \tag{5.6}$$

Proof. We prove (5.6) by induction on n. Clearly, (5.6) is true for both $n = 1$ and $n = 2$. Assume that $n \geq 3$, and that (5.6) is true when n is replaced by any l with $1 \leq l \leq n-1$, i.e.,

$$A_k^l \leq A_0^{l-k} A_l^k \quad \text{for } k = 0, \dots, l \text{ and } l = 1, \dots, n-1. \tag{5.7}$$

By (5.5),

$$A_{n-1}^2 \leq A_{n-2} A_n.$$

Thus,

$$A_{n-1}^{2(n-1)} \leq A_{n-2}^{n-1} A_n^{n-1}.$$

This, together with the inequality in (5.7) with $l = n-1$ and $k = n-2$, leads to

$$A_{n-1}^{2(n-1)} \leq A_0 A_{n-1}^{n-2} A_n^{n-1}.$$

Hence,

$$A_{n-1}^n \leq A_0 A_n^{n-1}. \tag{5.8}$$

By the inequality in (5.7) with $l = n-1$, we have

$$A_k^{n(n-1)} \leq A_0^{n(n-1-k)} A_{n-1}^{nk} \quad \text{for } k = 0, \dots, n-1.$$

This and (5.8) imply the inequality in (5.6) for $0 \leq k \leq n-1$. For $k = n$, the inequality in (5.6) is trivially true. **Q.E.D.**

Proof of Theorem 5.1. It follows from (1.10) and (5.1) that the spatial mean of the solution is in fact a constant: $\bar{h} = $ constant. Thus, by Definition 4.1, $h - \bar{h}$ is also a weak solution (with a different initial value) of the initial-boundary-value problem (4.1)–(4.3) on $[0, T]$ for any $T > 0$. Consequently, it follows from (5.1), Part (2) of Definition 4.1, and (2.5) that (see [20, 27])

$$\begin{aligned} \frac{d}{dt}[w_h(t)]^2 &= 2 ⨍_Q \left[h(x,t) - \bar{h} \right] \frac{\partial}{\partial t} \left[h(x,t) - \bar{h} \right] dx \\ &= -2 ⨍_Q W_m(h)\, dx + 2 ⨍_Q \frac{|\nabla h|^2}{1 + |\nabla h|^2}\, dx \leq 2 \qquad \forall\, t > 0. \end{aligned} \tag{5.9}$$

Integrating from t_0 to $t > t_0$ and taking the square root, we then obtain (5.2).

Set for any $t > t_0$

$$A_k = \sqrt{\fint_{t_0}^{t} \fint_Q W_k\left(h(x,\tau)\right) dx d\tau}, \qquad k = 0, \dots, m.$$

If $A_k = 0$ for some k with $0 \le k \le m$, then by (2.3), $h(x,\tau) = 0$ for a. e. $\tau \in (t_0, t)$; and hence all $A_j = 0$ $(0 \le j \le m)$. In this case, (5.3) holds true trivially.

Assume now all $A_k > 0$ $(0 \le k \le m)$. By (5.1) and (5.2), we have

$$A_0^2 = \fint_{t_0}^{t} [w_h(\tau)]^2 d\tau \le \fint_{t_0}^{t} \left[2(\tau - t_0) + [w_h(t_0)]^2\right] d\tau \le t - t_0 + [w_h(t_0)]^2. \tag{5.10}$$

Moreover, it follows from (5.9) that

$$\frac{1}{2}\frac{d}{dt}[w_h(t)]^2 + \fint_Q W_m(h(x,t))\, dx = \fint_Q \frac{|\nabla h|^2}{1+|\nabla h|^2} dx \le 1.$$

Thus, we have for any $t > t_0$ that

$$A_m^2 = \fint_{t_0}^{t} \fint_Q W_m(h(x,\tau))\, dx d\tau \le 1 + \frac{1}{2(t-t_0)}\left([w_h(t_0)]^2 - [w_h(t)]^2\right). \tag{5.11}$$

Fix integers k and p with $1 \le k \le m$ and $1 \le p \le \min(k, m-k)$. By (1.11) and integration by parts, we have the following: if k is even, then

$$\begin{aligned} A_k^2 &= \fint_{t_0}^{t} \fint_Q |\Delta^{k/2} h(x,\tau)|^2 dx d\tau \\ &= \begin{cases} (-1)^p \fint_{t_0}^{t} \fint_Q \Delta^{(k-p)/2} h(x,\tau)\, \Delta^{(k+p)/2} h(x,\tau)\, dx d\tau & \text{if } p \text{ is even}, \\ (-1)^p \fint_{t_0}^{t} \fint_Q \nabla\Delta^{(k-p-1)/2} h(x,\tau) \cdot \nabla\Delta^{(k+p-1)/2} h(x,\tau)\, dx d\tau & \text{if } p \text{ is odd}; \end{cases} \end{aligned}$$

if k is odd, then

$$\begin{aligned} A_k^2 &= \fint_{t_0}^{t} \fint_Q |\nabla\Delta^{(k-1)/2} h(x,\tau)|^2 dx d\tau \\ &= \begin{cases} (-1)^p \fint_{t_0}^{t} \fint_Q \nabla\Delta^{(k-p-1)/2} h(x,\tau) \cdot \nabla\Delta^{(k+p-1)/2} h(x,\tau)\, dx d\tau & \text{if } p \text{ is even}, \\ (-1)^p \fint_{t_0}^{t} \fint_Q \Delta^{(k-p)/2} h(x,\tau)\, \Delta^{(k+p)/2} h(x,\tau)\, dx d\tau & \text{if } p \text{ is odd}. \end{cases} \end{aligned}$$

In both cases, we have by the Cauchy-Schwarz inequality that

$$A_k^2 \le A_{k-p} A_{k+p}.$$

Consequently, by Lemma 5.1, (5.10), and (5.11), we obtain (5.3).

If $t > t_0 + [w_h(t_0)]^2$, then

$$\frac{[w_h(t_0)]^2}{2(t-t_0)} \le \frac{1}{2} \qquad \text{and} \qquad t - t_0 + [w_h(t_0)]^2 \le 2(t-t_0). \tag{5.12}$$

Since $-\log(\cdot)$ is a convex function, we obtain by Jensen's inequality, (5.3), (5.12), and the fact that $m \ge 2$ that

$$\begin{aligned}
⨍_{t_0}^{t} E(h(\tau))\, d\tau &\ge -\frac{1}{2}\log\left(1 + ⨍_{t_0}^{t} ⨍_Q |\nabla h(x,\tau)|^2 dx d\tau\right) \\
&\ge -\frac{1}{2}\log\left(1 + \left(1 + \frac{[w_h(t_0)]^2}{2(t-t_0)}\right)^{1/m} \left(t - t_0 + [w_h(t_0)]^2\right)^{(m-1)/m}\right) \\
&\ge -\frac{1}{2}\log\left(1 + 2^{(m-2)/m} 3^{1/m} (t-t_0)^{(m-1)/m}\right),
\end{aligned}$$

proving (5.4). **Q.E.D.**

The following is a direct consequence of Theorem 5.1 and the change of variables (1.7); it gives the precise dependence of the upper bounds on the material parameters m, F, M_m, S_c, and σ; see Section 6 for more discussions on this result:

Corollary 5.1 *Let $h : [0,\infty) \to H^m_{per}(Q)$ be a weak solution of the initial-boundary-value problem* (1.6), (4.2), *and* (4.3) *on $[0,T]$ for any $T > 0$. Let $t_0 \ge 0$. We have*

$$w_h(t) \le \sqrt{\frac{2FS_c}{\alpha^2}(t-t_0) + [w_h(t_0)]^2} \qquad \forall\, t \ge t_0, \tag{5.13}$$

$$\sqrt{⨍_{t_0}^{t} ⨍_Q W_k(h(x,\tau))\, dx d\tau} \le \frac{\sqrt{FS_c}}{\alpha M_m^{k/(2m)}} \left(1 + \frac{\alpha^2 [w_h(t_0)]^2}{2FS_c(t-t_0)}\right)^{\frac{k}{2m}} \left(t - t_0 + \frac{\alpha^2 [w_h(t_0)]^2}{FS_c}\right)^{\frac{m-k}{2m}}$$
$$\forall\, t > t_0,\ k = 1, \ldots, m. \tag{5.14}$$

6 Discussions

We first discuss two aspects of our analysis: the energy minimization and bounds for scaling laws. These are in fact general issues in the understanding of coarsening dynamics of energy-driven systems.

1. Energy minimization. The characterization of minimum energy and the magnitude of minimizers can help predict the dynamic scaling for the saturation of interface width. Assume in general an energy-driven, coarsening system saturates when a global energy-minimizer is reached. Then, it follows from Corollary 2.1 and (1.7) that the saturation interface width $w_s = w_s(l)$ with l being the linear size of the system is given by the scaling

$$w_s(l) \sim \frac{1}{\alpha}\sqrt{\frac{FS_c}{M_m}}\, l^m \tag{6.1}$$

with the *roughness exponent* m. More generally, the saturated, time and space averaged, k-th gradient of the surface height scales as

$$\frac{1}{\alpha}\sqrt{\frac{FS_c}{M_m}}\, l^{m-k} \qquad (1 \le k \le m).$$

Notice that the prefactor in these scaling laws is the square root of the ratio of FS_c and M_m. Thus, the surface roughness is uniquely determined by m and the competition of the deposition, ES effect, and surface relaxation. Both the deposition and ES effect make a surface rough and the relaxation makes a surface smooth.

Now, the saturation time $t_s = t_s(l)$ can be regarded as the time at which a global minimizer is reached in the dynamics. Thus, the interface width at t_s is equal to that of a global minimizer h which in turn is equal to the saturation interface width: $w_h(t_s) = w_s(l)$. This, together with the upper bound (5.13) and the scaling (6.1), leads to a lower bound for t_s in the *dynamic scaling*

$$t_s(l) \ge \frac{1}{2M_m} l^{2m} \tag{6.2}$$

with the *dynamics exponent* $2m$. This indicates that the dynamic scaling is determined only by m and the mobility M_m.

The concept of λ-minimizers arises naturally from the following simple scenario of the coarsening dynamics of an energy-driven system:

(1) There exist a sequence of λ-minimizers that are equilibriums. The wavelength of these minimizers, $\lambda_N, \ldots, \lambda_1$, increases, and the corresponding energy decreases. The largest wavelength λ_1 is the characteristic wavelength of a global minimizer. It may or may not be of the order of the linear size of an underlying system;

(2) The system is always near one of such equilibrium;

(3) The system moves to the next equilibrium with larger wavelength to reduce the energy.

Our analysis in Section 2 shows the existence of a sequence of λ-minimizer with the desired properties for our underlying system. In general, it remains challenging to understand mathematically how stable a λ-minimizer is and how much time is needed for a system to move from one λ-minimizer to another.

2. Upper bounds for scaling laws. By Corollary 5.1, our upper bound for the interface width always scales as $t^{1/2}$, independent of m. This agrees with the early analysis in [10]. Our result also indicates that prefactor of this scaling depends only on FS_c, cf. (5.13). Taking $k = 1$ in (5.14), we see that the surface slope scales as

$$\frac{\sqrt{FS_c}}{\alpha M_m^{1/(2m)}} t^{(m-1)/(2m)}.$$

This and (5.13) indicate that the characteristic lateral size of mounds $\lambda(t)$ should scale as

$$\lambda(t) \sim \sqrt{2} M_m^{1/(2m)} t^{1/(2m)} \tag{6.3}$$

with the *coarsening exponent* $1/(2m)$. Notice that the prefactor in this scaling only depends on the mobility M_m.

Further studies are needed to obtain an upper bound for the scaling law (6.3) of $\lambda(t)$, the characteristic lateral size of mounds; and to show the optimality of all bounds.

We remark that our studies on the growth scaling laws agree with previous studies, both analytical and numerical [10,12,25,27,28,31]. In particular, we recover the analytical results in [10] for all $m \geq 2$ that are obtained under a strong assumption on scaling (cf. Eq. (8) of [10]).

We now compare our results with experiments. For $m = 2$, the case of surface diffusion, our predictions of the scaling laws agree well with the following reported experiments: (1) The growth of Cu film at temperature $200K$ for which the ES effect is believed to be strong. This is reported in [8] (cf. FIG. 1 and paragraph 1 of page 3 in [8]); (2) The epitaxial growth of Fe(001) films on Mg(001) substrate at the substrate temperate 400–$450K$ in which the ES effect gives rise to pyramid-like surface structure. This is reported in [35] (cf. FIG. 3 and the discussion at the end of paragraph 2 of page 3 in [35]).

Experiments reported in [33] on the homoepitaxy of Fe(001) at room temperature show that the coarsening exponent is 0.16 ± 0.04, close to $1/6$. This is predicted by our analysis with the case $m = 3$, cf. (6.3) with $m = 3$. In [33], a coarse-grained model is proposed (cf. Eq. (2) in [33]); and numerical calculations based on this model are also reported. These calculations reproduce the $t^{1/6}$ scaling of coarsening from the experiment. In this model, the high-order relaxation term is exactly the term (1.2) with $m = 3$. The next two terms in this model describe the ES effect when the surface gradient is small.

This can be seen through an expansion of the low-order term in the effective energy:

$$-\log\left(1+|\nabla h|^2\right) = \frac{1}{2}\left(|\nabla h|^2 - 1\right)^2 - \frac{1}{2} + O\left(|\nabla h|^6\right) \qquad \text{if } |\nabla h| \ll 1.$$

The last term in the model describes the up-down asymmetry. As pointed out in [33] (cf. paragraph 2 of the last page of [33]), the agreement between numerical calculations and experiment on the coarsening rate is: "insensitive to the presence or absence of the symmetry-breaking term." Therefore, the experimentally observed coarsening rate results expectedly from the competition between the high-order surface relaxation (1.2) with $m = 3$ and the ES effect. It remains challenging to understand the kinetic origin of such a relaxation mechanism.

We wonder if our analysis can provide some insight for experiments. For instance, is it possible experimentally to measure the physical parameters such as the mobility M_m or the ES parameter S_c, using our predicted scaling laws such as (5.13), (5.14), (6.1), and (6.2)?

Finally, we discuss a natural extension of the current (1.2) to a linear combination of several high-order relaxation terms

$$\mathbf{j}_{RE} = \sum_{m=2}^{p} (-1)^m M_m \nabla \Delta^{m-1} h$$

for some integer $p \geq 3$. Models with terms of high-order derivatives like this has been used in small-slope approximations of anisotropic surface free-energy density [21, 32].

Our methods can be used to analyze the corresponding effective energy functionals. In particular, for the corresponding re-scaled, singularly perturbed functionals (cf. (2.7) and (2.8)), global minimizers exist and the minimum energy scales as $O(\log \varepsilon)$. But, the exact constant in this asymptotics is not immediately clear. Similarly, the gradients of any global minimizer are inversely proportional to ε. But, the exact two-sided bounds for all the gradients, as in Part (2) of Theorem 2.1, may no longer hold true with a single parameter p.

Our argument (5.2) can be used directly to obtain an upper bound for the $t^{1/2}$ of interface width. Since all the derivative terms can be controlled by the terms with the highest order derivatives, the energy method we use in obtaining bounds for gradients can be applied directly to the extended model. In particular, we expect the estimates (5.3), with m replaced by p, to hold true for the new model. However, the constants in these bounds may not be simple in forms.

Acknowledgments

This work was partially supported by the US National Science Foundation through grant DMS-0451466 and by the US Department of Energy through grant DE-FG02-05ER25707. The author thanks referees for helpful comments and suggestions.

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