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In the proof of Theorem 2.1 in Section 4, Jensen’s inequality was used incorrectly, cf. the first inequality in line 7 on page 2550. To correct this, we define \( K := \{ u \in H^1(\Omega) : u = \psi_0 \text{ on } \partial \Omega \} \) and define \( J : K \to \mathbb{R} \cup \{ +\infty \} \) as in line 11 on page 2550. By the second paragraph of page 2550, the functional \( J[\cdot] \) has a minimizer \( v \in K \). We now prove \( v \) is bounded on \( \Omega \). Hence \( \chi_{\Omega_s} B \left( v + G - \hat{\psi}_0 / 2 \right) \in L^2(\Omega_s) \). After this, the rest of the proof, starting from the third paragraph on page 2550, remains the same.

Let \( \lambda > \| \psi_0 \|_{L^\infty(\Omega)} \). Define \( v_\lambda : \Omega \to \mathbb{R} \) by \( v_\lambda(x) = v(x) \) if \( |v(x)| \leq \lambda \), \( v_\lambda(x) = \lambda \) if \( v(x) > \lambda \), and \( v_\lambda(x) = -\lambda \) if \( v(x) < -\lambda \), where \( x \in \Omega \). Clearly \( v_\lambda \in K \) and thus \( J[v] \leq J[v_\lambda] \). Consequently, since \( |\nabla v_\lambda| \leq |\nabla v| \) a.e. \( \Omega \),

\[
\int_{\Omega_s} B \left( v(x) + G(x) - \hat{\psi}_0(x) / 2 \right) dx \leq \int_{\Omega_s} B \left( v_\lambda(x) + G(x) - \hat{\psi}_0(x) / 2 \right) dx. \tag{1}
\]

The definition of \( B : \mathbb{R} \to \mathbb{R} \) is given in (1.14) on page 2540. For the case of point ions, \( B \) is clearly convex and \( B'(s) \to \pm\infty \) as \( s \to \pm\infty \). For the case of finite-size ions, we have

\[
B'(s) = -\frac{\sum_{j=1}^{M} c_j \infty q_j e^{-\beta q_j s}}{1 + a^3 \sum_{j=1}^{M} c_j \infty e^{-\beta q_j s}}
\]

\[
\left( 1 + a^3 \sum_{j=1}^{M} c_j \infty e^{-\beta q_j s} \right)^2 B''(s) = \beta \sum_{j=1}^{M} c_j \infty q_j^2 e^{-\beta q_j s}
\]

\[
+ \beta a^3 \left[ \left( \sum_{j=1}^{M} c_j \infty q_j^2 e^{-\beta q_j s} \right) \left( \sum_{j=1}^{M} c_j \infty e^{-\beta q_j s} \right) - \left( \sum_{j=1}^{M} c_j \infty q_j e^{-\beta q_j s} \right)^2 \right].
\]

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The Cauchy–Schwarz inequality implies that $B''(s) > 0$. Hence $B$ is also convex. Moreover, using the electrostatic neutrality condition $\sum_{j=1}^{M} c_{j}q_{j} = 0$, we have
\[
\lim_{s \to \infty} B'(s) = -a^{-3} \min_{1 \leq j \leq M} q_{j} > 0 \quad \text{and} \quad \lim_{s \to -\infty} B'(s) = -a^{-3} \max_{1 \leq j \leq M} q_{j} < 0.
\]

It then follows from (1) of this note, the definition of $v_{\lambda}$, and the fact that both $G$ and $\hat{\psi}$ are bounded on $\Omega_{s}$ that there exists $\alpha > 0$ such that for $\lambda > 0$ enough,
\[
0 \geq \int_{\{x \in \Omega_{s} : v(x) > \lambda\}} \left[ B \left( v(x) + G(x) - \frac{\hat{\psi}(x)}{2} \right) - B \left( \lambda + G(x) - \frac{\hat{\psi}(x)}{2} \right) \right] dx \\
+ \int_{\{x \in \Omega_{s} : v(x) < -\lambda\}} \left[ B \left( v(x) + G(x) - \frac{\hat{\psi}(x)}{2} \right) - B \left( -\lambda + G(x) - \frac{\hat{\psi}(x)}{2} \right) \right] dx \\
\geq \int_{\{x \in \Omega_{s} : v(x) > \lambda\}} B' \left( \lambda + G(x) - \frac{\hat{\psi}(x)}{2} \right) [v(x) - \lambda] dx \\
+ \int_{\{x \in \Omega_{s} : v(x) < -\lambda\}} B' \left( -\lambda + G(x) - \frac{\hat{\psi}(x)}{2} \right) [v(x) + \lambda] dx \\
\geq \alpha \int_{\{x \in \Omega_{s} : v(x) > \lambda\}} [v(x) - \lambda] dx - \alpha \int_{\{x \in \Omega_{s} : v(x) < -\lambda\}} [v(x) + \lambda] dx \\
= \alpha \int_{\{x \in \Omega_{s} : |v(x)| > \lambda\}} [|v(x)| - \lambda] dx \\
\geq 0.
\]
Hence the last integral is 0. This implies that the measure of the set $\{x \in \Omega_{s} : |v(x)| > \lambda\}$ is 0. Therefore $|v| \leq \lambda$ a.e. $\Omega_{s}$.

**Remark.** By Theorem 2.4 and Theorem 2.5, there exists a unique set of equilibrium ionic concentrations $c = (c_{1}, \ldots, c_{M})$ for both the case of point ions and that of finite-size ions. These concentrations are bounded and are related to the corresponding electrostatic potential $\psi$ by the Boltzmann distributions (1.9). The potential $\psi$ is bounded in $\Omega_{s}$, cf. (3.11). It is a solution to the underlying boundary-value problem of Poisson–Boltzmann equation, cf. (1.4), (1.8) and (1.9). This provides a different but indirect proof of the existence of a bounded solution to the boundary-value problem of the Poisson–Boltzmann equation.

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