The Immersed Interface Method for Elasticity Problems with Interfaces*

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Abstract. An immersed interface method for solving linear elasticity problems with two phases separated by an interface has been developed in this paper. For the problem of interest, the underlying elasticity modulus is a constant in each phase but vary from phase to phase. The basic goal here is to design an efficient numerical method using a fixed Cartesian grid. The application of such a method to problems with moving interfaces driving by stresses has a great advantage: no re-meshing is needed. A local optimization strategy is employed to determine the finite difference equations at grid points near or on the interface. The bi-conjugate gradient method and the GMRES with preconditioning are both implemented to solve the resulting linear systems of equations and compared. Numerical results are presented to show that the method is second-order accurate.

Keywords: elasticity, interfaces, jump conditions, finite differences, the immersed interface method.

2000 AMS Mathematics Subject Classification: 65N06, 74B10, 74S30.

1 Introduction

Elasticity problems of multiple phase elastic materials separated by phase interfaces often arise in materials science [18, 21]. Two important examples of such problems occur in the microstructural evolution of precipitates in an elastic matrix due to the diffusion of concentration and in the morphological instability due to stress-driven surface diffusion in solid thin films, cf. e.g., [2, 10, 13] and the references therein. The understanding of these physical processes is crucial to improve material stability properties, and in turn to develop new and advanced materials that have applications in automobile manufacture, aircraft industries, and modern communication technologies. However, solving such elasticity problems are often very difficult due to complicated geometries, multiple components, and nonlinearities that appear in these problems. For these reasons, there has been a great interest recently, in all materials science, scientific computing, and applied mathematics communities, in developing efficient and accurate numerical techniques for elasticity problems with interfaces separating multiple phases.

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In this paper, we consider a two-dimensional problem that can arise from many modeling situations such as two-phase elastic plates in the setting of plane strain or plane stress [5, 9, 20]. We assume that the elastic material occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma_0 = \partial\Omega$. For our computational purpose, we assume that the domain Ω is rectangular. The two phases of the material occupy regions Ω^+ and Ω^-, respectively, so $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$ and $\Omega^+ \cap \Omega^- = \emptyset$. We denote by $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ the interface separating these two phases, cf. Figure 1.1.

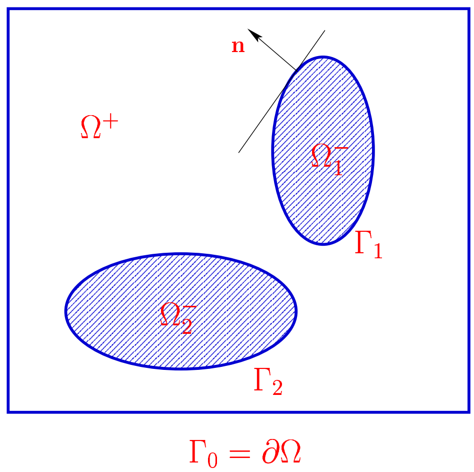

Figure 1.1: The geometry of the underlying problem.

The equilibrium equation, interface conditions on Γ, and the boundary conditions on Γ_0 [5, 9, 18, 20] are:

$$\nabla \cdot \sigma + \mathbf{F} = 0 \quad \text{in } \Omega^+ \cup \Omega^-, \tag{1.1}$$

$$[\mathbf{u}] = 0 \quad \text{on } \Gamma, \tag{1.2}$$

$$[\sigma \mathbf{n}] = \mathbf{T} \quad \text{on } \Gamma, \tag{1.3}$$

$$\mathbf{u} = \mathbf{u_0} \quad \text{on } \Gamma_0, \tag{1.4}$$

where σ is the stress tensor, $\mathbf{F} = (F_1, F_2)^T : \Omega \to \mathbb{R}^2$ is the body force which is known (a superscript T denotes the transpose), $\mathbf{u} = (u_1, u_2)^T : \Omega \to \mathbb{R}^2$ with $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ is the displacement field, for a function v, $[v] = v^+ - v^-$ with $v^\pm = v|_{\Omega^\pm}$ denotes the jump of v across the interface, $\mathbf{n} = (n_1, n_2)^T$ is the unit normal vector to the interface Γ, pointing from the $-$ phase to the $+$ phase, $\mathbf{T} = (\phi, \psi)^T$ is a given vector-valued function on the interface Γ which measures the jump of the traction $\sigma\mathbf{n}$ across the interface Γ, and $\mathbf{u_0}$ is a given, vector-valued function that represents the displacement on the boundary Γ_0. The jump condition (1.2) means that $\mathbf{u}$ is continuous across the interface. It indicates that the underlying material has no fracture. Notice that we have introduced a generally nonzero data $\mathbf{T}$ for the jump of the traction in (1.3). This will be useful to model more general physical situations.

We assume that the material is isotropic. So, in the setting of plane deformation, the stress-strain relation is given by

$$\sigma = \lambda \mathrm{tr}\,(\varepsilon) I + 2\mu\varepsilon$$

$$= \begin{bmatrix} (\lambda+2\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y} & \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \lambda\frac{\partial u_1}{\partial x} + (\lambda+2\mu)\frac{\partial u_2}{\partial y} \end{bmatrix}, \tag{1.5}$$

where

$$\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{bmatrix} \tag{1.6}$$

is the linear strain, I is the 2×2 identity tensor,

$$\mu = \frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

are the Lamé constants, E is Young's modulus, and ν is Poisson's ratio.

In modeling a two-phase elastic material, we assume that all the material parameters μ, λ, E, and ν are piecewise constants. In particular, we assume that the shear modulus and Poisson's ratio are given, respectively, by

$$\mu = \begin{cases} \mu^+ & \text{in } \Omega^+ \\ \mu^- & \text{in } \Omega^- \end{cases} \qquad \text{and} \qquad \nu = \begin{cases} \nu^+ & \text{in } \Omega^+ \\ \nu^- & \text{in } \Omega^- \end{cases},$$

where all μ^+, μ^-, ν^+, ν^- are positive constants. Usually, Poisson's ratio $\nu^+, \nu^- \in (0, 0.5)$ (cf. [1, 11]).

We let $f = -F_1/(\mu+\lambda)$ and $g = -F_2/(\mu+\lambda)$, and equations (1.1) and (1.3) can be written as

$$2(1-\nu)\frac{\partial^2 u_1}{\partial x^2} + (1-2\nu)\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial x \partial y} = f(x,y), \tag{1.7}$$

$$(1-2\nu)\frac{\partial^2 u_2}{\partial x^2} + 2(1-\nu)\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_1}{\partial x \partial y} = g(x,y), \tag{1.8}$$

$$\left[\frac{2\mu}{1-2\nu}\left((1-\nu)\frac{\partial u_1}{\partial x} + \nu\frac{\partial u_2}{\partial y}\right)n_1 + \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)n_2\right]\Big|_\Gamma = \phi, \tag{1.9}$$

$$\left[\mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)n_1 + \frac{2\mu}{1-2\nu}\left(\nu\frac{\partial u_1}{\partial x} + (1-\nu)\frac{\partial u_2}{\partial y}\right)n_2\right]\Big|_\Gamma = \psi, \tag{1.10}$$

where $[\cdot]$ is defined as the jump of the quantity between the outside and the inside of the interface.

There exist several numerical methods for solving general elasticity problems that do not involve interfaces. Among them, the finite element method and the boundary integral or boundary element method appear to be very successful, cf. e.g., [3, 19, 23] and the references therein. However, in treating moving interface problems, the use of fixed Cartesian grids often shows advantages in practical computations [12]. It is therefore desirable to develop new, efficient methods based on finite difference schemes on fixed Cartesian grids for our underlying elasticity problems with interfaces. This is our primary goal of the work. Our idea is to use the immersed interface method [22, 15, 16] to derive a finite difference scheme for the elasticity problem with an interface. This is a natural strategy, since the geometrical complexity of the problem is local.

The curved interface in the underlying problem brings up several substantial difficulties in the development and analysis of numerical schemes.

- **Discretization.** With a uniform Cartesian grid, the interface is typically not aligned with the grid but rather cuts between grid points. Thus, for grid points near the interface, the stencil of a standard finite difference scheme will contain points from both sides of the interface. Due to the non-smoothness of u_1 and u_2, differentiating u_1 and u_2 across the interface using standard finite difference schemes will not produce accurate approximations to the derivatives of u_1 and u_2. The cross derivative terms in the differential equations need special treatments in the discretization on the grid points near or on the interface.

- **Solving the system of discrete linear equations.** Because of the presence of interfaces and non-smoothness in the solution, the system of discrete equations is no longer symmetric or positive definite. The structure of the system makes it hard to find an efficient solver.

- **Error analysis.** Due to the complexity of interfaces and the non-smoothness in the solutions, it is difficult to perform convergence analysis in the conventional way.

Our main contribution of this paper is the development of the immersed interface method (IIM) to the underlying elasticity problems with interfaces. The key in our method is to utilize the local coordinate transformation to carefully analyze the relations of quantities from one side to the another. Such relations shall lead us to find accurate finite difference schemes for grid points near or on the interface.

The paper is organized as follows. In Section 2, we describe our algorithm. In Section 3, we introduce local coordinate transformation that is essential to the development of our method. In Section 4, we describe interface relations. In Section 5, we derive the finite difference scheme for irregular grid points. The linear solvers are also discussed. In Section 6, we present our numerical results. Finally, in Section 7, we draw conclusions.

2 Description of the Algorithm

For simplicity, we assume that the domain Ω is a square: $\Omega = (a, b) \times (c, d)$ with $d - c = b - a$. Let $n \geq 1$ be an integer and $h = (b - a)/n = (d - c)/n$. Let

$$x_i = a + ih, \ y_j = c + jh, \qquad i, j = 0, \cdots, n.$$

We wish to solve the problem using a finite difference method on the uniform Cartesian grid. Our result will be a finite difference scheme of the form

$$\left\{ \begin{array}{lcl} \sum\limits_k \alpha_k (U_1)_{i+i_k, j+j_k} + \sum\limits_k \beta_k (U_2)_{i+i_k, j+j_k} & = & f_{ij} + C^1_{ij}, \\ \sum\limits_k \gamma_k (U_1)_{i+i_k, j+j_k} + \sum\limits_k \delta_k (U_2)_{i+i_k, j+j_k} & = & g_{ij} + C^2_{ij}, \end{array} \right. \tag{2.1}$$

at any grid point (x_i, y_j), where $(U_1)_{i,j}$ and $(U_2)_{i,j}$ approximate $u_1(x_i, y_j)$ and $u_2(x_i, y_j)$, respectively, $f_{ij} = f(x_i, y_j)$, $g_{ij} = g(x_i, y_j)$, and all α_k, β_k, γ_k, and δ_k are undetermined coefficients. Each sum over k in (2.1) involves only finite number of grid points that are centered at (x_i, y_j) (at most nine grid points are involved in our algorithm), all $i_k, j_k \in \{-1, 0, 1\}$. The coefficients α_k, β_k, γ_k, and δ_k and the indices i_k, j_k will depend on (i, j). We omit the dependency for simplicity. By finding these coefficients properly, we can obtain a second-order accurate finite difference scheme.

2.1 Classification of grid points

Before we explain the finite difference scheme, we classify all grid points into two categories: *regular* and *irregular*. We say a grid point (x_i, y_j) is *regular* (Figure 2.1 (a)) if the interface Γ does not cut through any points in the standard nine point stencil centered at (x_i, y_j). We say a grid point is *irregular*, if it is not regular. An irregular grid point is further classified as type I, if the interface crosses the five point stencil centered at this point (Figure 2.1 (b)), and type II, if otherwise (Figure 2.1 (c) and (d)).

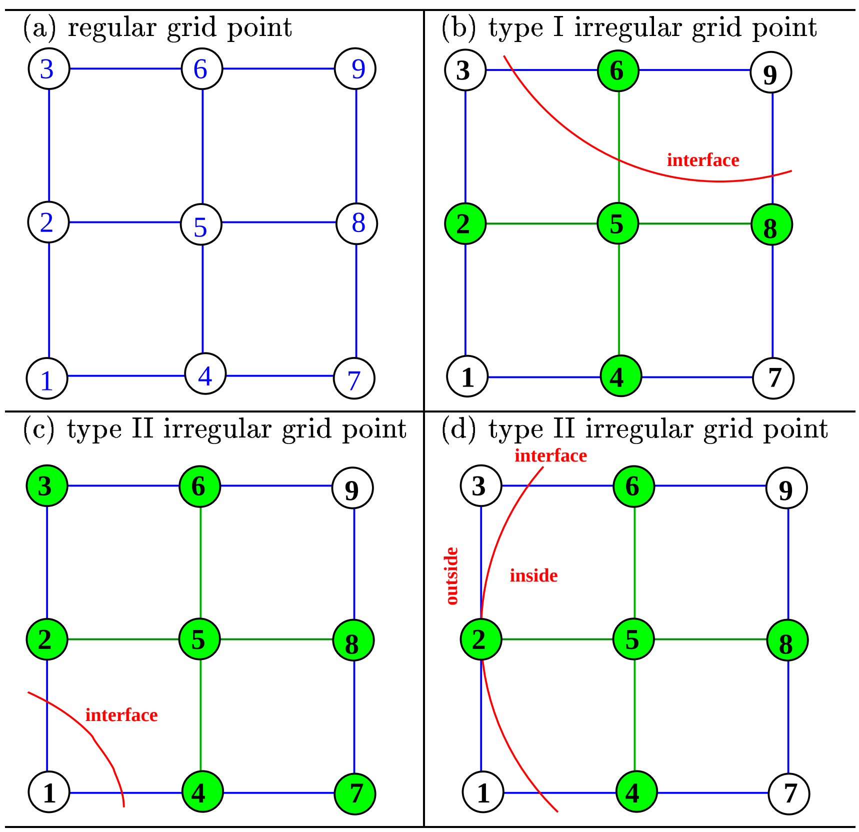

Figure 2.1: Classification of grid points.

At a regular grid point (Figure 2.1 (a)), we use the standard central finite difference scheme. At an irregular grid point (see Figure 2.1 (b)–(d)), we derive a finite difference scheme according to how the interface cuts through the five-point stencil.

2.2 Discretization at regular grid points

At a regular grid point (x_i, y_j), we have the following approximations of the second-order partial derivatives for a given smooth function u

$$\begin{aligned} u_{xx}(x_i, y_j) &= \frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{h^2} + \mathcal{O}(h^2), \\ u_{yy}(x_i, y_j) &= \frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{h^2} + \mathcal{O}(h^2), \\ u_{xy}(x_i, y_j) &= \frac{u(x_{i+1}, y_{j+1}) + u(x_{i-1}, y_{j-1}) - u(x_{i-1}, y_{j+1}) - u(x_{i+1}, y_{j-1})}{4h^2} + \mathcal{O}(h^2), \end{aligned}$$

cf. Figure 2.2. These lead to the following discretization of (1.7) and (1.8):

$$\begin{aligned} &\frac{2(1-\nu)}{h^2}\Big((U_1)_{i-1,j} + (U_1)_{i+1,j}\Big) + \frac{1-2\nu}{h^2}\Big((U_1)_{i,j-1} + (U_1)_{i,j+1}\Big) \\ &\quad -\frac{2(3-4\nu)}{h^2}(U_1)_{i,j} + \frac{1}{4h^2}\Big((U_2)_{i+1,j+1} + (U_2)_{i-1,j-1} \\ &\quad -(U_2)_{i-1,j+1} - (U_2)_{i+1,j-1}\Big) = f_{ij}, \\ &\frac{1-2\nu}{h^2}\Big((U_2)_{i-1,j} + (U_2)_{i+1,j}\Big) + \frac{2(1-\nu)}{h^2}\Big((U_2)_{i,j-1} + (U_2)_{i,j+1}\Big) \\ &\quad -\frac{2(3-4\nu)}{h^2}(U_2)_{i,j} + \frac{1}{4h^2}\Big((U_1)_{i+1,j+1} + (U_1)_{i-1,j-1} \\ &\quad -(U_1)_{i-1,j+1} - (U_1)_{i+1,j-1}\Big) = g_{ij}, \end{aligned} \tag{2.2}$$

where $(U)_{i,j}$ stands for the approximation of $u(x_i, y_j)$, and in this case $C^1_{ij} = C^2_{ij} = 0$, cf. (2.1).

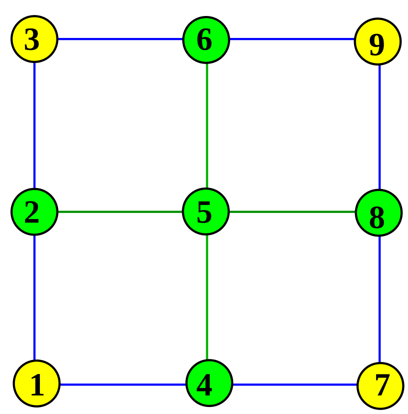

Figure 2.2: Discretization at a regular grid point.

2.3 Discretization at type I irregular grid points

Following [15] and [16], we use a nine-point stencil for u_1 and u_2 in both equations of (2.1) at such a grid point (x_i, y_j) (cf. Figure 2.1 (b)). To determine the coefficients of the finite difference equations, we first choose a point (x_i^*, y_j^*) on the interface Γ that is near (x_i, y_j). We replace $(U_1)_{i+i_k,j+j_k}$ and $(U_2)_{i+i_k,j+j_k}$ with the exact solution $u_1(x_{i+i_k}, y_{j+j_k})$ and $u_2(x_{i+i_k}, y_{j+j_k})$ in (2.1) and use the Taylor expansion at (x_i^*, y_j^*) for each term in order to set up a system of equations for the coefficients. Since the solution from both sides is involved, we use the superscript $-$ and $+$ to denote the limiting values of a function from one side or the other. For example, if the point (x_i, y_j) is located inside the interface, then we can expand $u(x_i, y_j)$ to get

$$\begin{aligned} u(x_i, y_j) &= u^- + u_x^-(x_i - x_i^*) + u_y^-(y_j - y_j^*) + \frac{1}{2}u_{xx}^-(x_i - x_i^*)^2 \\ &\quad + \frac{1}{2}u_{yy}^-(y_j - y_j^*)^2 + u_{xy}^-(x_i - x_i^*)(y_j - y_j^*) + \mathcal{O}(h^3). \end{aligned}$$

If we do such expansion at each grid point used in the finite difference equations in (2.1), then the local truncation errors T_{ij}^1 and T_{ij}^2 (for the first and second equation respectively) can be expressed as a linear combination of all the values $u_1^\pm$, $u_{1x}^\pm$, $u_{1y}^\pm$, $u_{1xx}^\pm$, $\cdots$, $u_{2yy}^\pm$, and $u_{2xy}^\pm$.

Next, we express all the values from $+$ side, u_1^+, u_{1x}^+, u_{1y}^+, $\cdots$, u_{2yy}^+, u_{2xy}^+, in terms of the values on the $-$ side, u_1^-, u_{1x}^-, u_{1y}^-, $\cdots$, u_{2xx}^-, u_{2yy}^-, u_{2xy}^-. To do so, we need to use the interface conditions,

$$u_1^+ = u_1^-, \qquad u_2^+ = u_2^-, \tag{2.3}$$

$$\begin{aligned} &\frac{2\mu^+}{1-2\nu^+}\Big((1-\nu^+)(u_1)_x^+ + \nu^+(u_2)_y^+\Big)n_1 + \mu^+\Big((u_1)_y^+ + (u_2)_x^+\Big)n_2 \\ &= \frac{2\mu^-}{1-2\nu^-}\Big((1-\nu^-)(u_1)_x^- + \nu^-(u_2)_y^-\Big)n_1 \\ &\quad + \mu^-\Big((u_1)_y^- + (u_2)_x^-\Big)n_2 + \phi, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\mu^+\Big((u_1)_y^+ + (u_2)_x^+\Big)n_1 + \frac{2\mu^+}{1-2\nu^+}\Big(\nu^+(u_1)_x^+ + (1-\nu^+)(u_2)_y^+\Big)n_2 \\ &= \mu^-\Big((u_1)_y^- + (u_2)_x^-\Big)n_1 + \frac{2\mu^-}{1-2\nu^-}\Big(\nu^-(u_1)_x^- \\ &\quad + (1-\nu^-)(u_2)_y^-\Big)n_2 + \psi. \end{aligned} \tag{2.5}$$

To obtain all the coefficients α_k, β_k, γ_k, and δ_k in the finite difference equations (2.1), more interface relations are needed. Differentiating the above equations and manipulating the results allow us to perform the desired elimination, as detailed in Section 4. In order to do so, it is very convenient to first perform a local coordinate transformation in the normal direction ξ, and the tangential direction η of Γ at (x_i^*, y_j^*).

Once the local truncation errors T_{ij}^1 and T_{ij}^2 are expressed as a linear combination of the values from just one side, say u_1^-, u_{1x}^-, u_{1y}^-, u_{1xx}^-, u_{1yy}^-, u_{1xy}^-, u_2^-, u_{2x}^-, u_{2y}^-, u_{2xx}^-, u_{2yy}^-, u_{2xy}^-, we must require that the coefficient of each of these terms to vanish in order to match the partial differential equation up to second order derivative terms. This gives us a system of twelve linear equations from the first and second finite difference equations respectively. Nine-point stencil will be used to obtain a solvable system, see Section 5 for all these derivations.

2.4 Discretization at type II irregular grid points

For a type II irregular grid point (cf. Figure 2.1 (c) and (d)), we use

- five-point stencil for u_1 and seven-point or four-point stencil for the cross derivative of u_2 in the first equation, and
- five-point stencil for u_2 and seven-point or four-point stencil for the cross derivative of u_1 in the second equation.

Therefore equations (1.7) and (1.8) have the following form:

$$\frac{2(1-\nu)}{h^2}\Big((U_1)_{i-1,j}+(U_1)_{i+1,j}\Big)+\frac{1-2\nu}{h^2}\Big((U_1)_{i,j-1}+(U_1)_{i,j+1}\Big)$$
$$-\frac{2(3-4\nu)}{h^2}(U_1)_{i,j}+\sum_k \beta_k (U_2)_{i+i_k,j+j_k}=f_{ij}, \quad (2.6)$$

$$\frac{1-2\nu}{h^2}\Big((U_2)_{i-1,j}+(U_2)_{i+1,j}\Big)+\frac{2(1-\nu)}{h^2}\Big((U_2)_{i,j-1}+(U_2)_{i,j+1}\Big)$$
$$-\frac{2(3-4\nu)}{h^2}(U_2)_{i,j}+\sum_k \gamma_k (U_1)_{i+i_k,j+j_k}=g_{ij}. \quad (2.7)$$

To determine the coefficients β_k and γ_k in (2.6) and (2.7), we use four-point stencil. For the grid points shown in Figures 2.1 (a) and (b), we can apply the following expansion for a smooth function u to derive our finite difference scheme

$$u_{xy} = \frac{u^{(1)}-u^{(2)}-u^{(4)}+u^{(5)}}{h^2}+\mathcal{O}(h) \quad (2.8)$$
$$= \frac{u^{(2)}-u^{(3)}-u^{(5)}+u^{(6)}}{h^2}+\mathcal{O}(h) \quad (2.9)$$
$$= \frac{u^{(4)}-u^{(5)}-u^{(7)}+u^{(8)}}{h^2}+\mathcal{O}(h) \quad (2.10)$$
$$= \frac{u^{(5)}-u^{(6)}-u^{(8)}+u^{(9)}}{h^2}+\mathcal{O}(h), \quad (2.11)$$

where $u^{(i)}$ denotes u value at the point numbered i in Figure 2.1. Which formula from (2.8) $\sim$ (2.11) to use depends on how the interface crosses the nine -point stencil. In the specific case in Figure 2.1 (d), we use the expansion (2.10) or (2.11), where points numbered (4) to (9) are in the same side of the interface. However, when one grid point is on the other side of the other eight grid points in the nine-point stencil as shown in Figures 2.3 (a) and (b), we apply the following formula for a smooth function u. In this case we have

$$u_{xy} = \frac{u^{(2)}-u^{(3)}+u^{(4)}-2u^{(5)}+u^{(6)}-u^{(7)}+u^{(8)}}{2h^2}+\mathcal{O}(h^2) \quad (2.12)$$
$$= \frac{u^{(1)}-u^{(2)}-u^{(4)}+2u^{(5)}-u^{(6)}-u^{(8)}+u^{(9)}}{2h^2}+\mathcal{O}(h^2). \quad (2.13)$$

Whether we use (2.12) or (2.13) depends on the geometry. We use, for example, the formula (2.12) in Figure 2.3 (a), and (2.13) in Figure 2.3 (b).

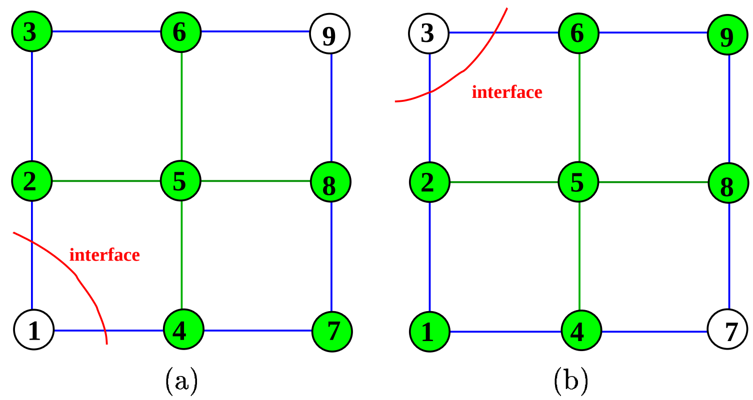

Figure 2.3: An illustration of the geometry of type II irregular grid points.

3 Transformations of Local Coordinates

For each type I irregular point (x_i, y_j), we need to find a point (x_i^*, y_j^*) on the interface. We usually take this point as the projection of (x_i, y_j) on the interface if the interface is smooth at this point. Otherwise we can take any point on the interface where the interface is smooth.

After (x_i^*, y_j^*) is selected, we apply the local coordinate transformation at this point. Let θ be the angle between the $x-$axis and the normal direction, pointing in the direction of the $+$ side, cf. Figure 3.1. The transformation is defined as follows:

$$\left\{ \begin{array}{lcl} \xi & = & (x - x_i^*)\cos(\theta) + (y - y_j^*)\sin(\theta), \\ \eta & = & -(x - x_i^*)\sin(\theta) + (y - y_j^*)\cos(\theta). \end{array} \right. \tag{3.1}$$

Under this local coordinate transformation, the governing equations in (1.7) and (1.8) become

$$\begin{aligned} &(\cos^2\theta + 1 - 2\nu)u_{1\xi\xi} - 2\sin\theta\cos\theta u_{1\xi\eta} + (\sin^2\theta + 1 - 2\nu)u_{1\eta\eta} \\ &+ \sin\theta\cos\theta u_{2\xi\xi} + (\cos^2\theta - \sin^2\theta)u_{2\xi\eta} - \sin\theta\cos\theta u_{2\eta\eta} = f, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &(\sin^2\theta + 1 - 2\nu)u_{2\xi\xi} + 2\sin\theta\cos\theta u_{2\xi\eta} + (\cos^2\theta + 1 - 2\nu)u_{2\eta\eta} \\ &+ \sin\theta\cos\theta u_{1\xi\xi} + (\cos^2\theta - \sin^2\theta)u_{1\xi\eta} - \sin\theta\cos\theta u_{1\eta\eta} = g, \end{aligned} \tag{3.3}$$

where $f = f(x, y) = \tilde{f}(\xi, \eta)$, $g = g(x, y) = \tilde{g}(\xi, \eta)$ for simplicity.

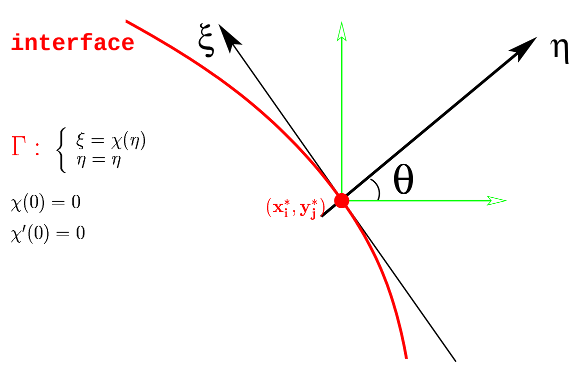

Figure 3.1: The local coordinate transformation.

4 Interface Relations

We consider a fixed point (x_i^*, y_j^*) and define a new ξ-η coordinate system based on the directions normal and tangential to Γ at this point using the formulas (3.1). In a neighborhood of this point, the interface can be parameterized as $\xi = \chi(\eta)$, $\eta = \eta$. Notice that $\chi(0) = 0$ and $\chi'(0) = 0$ provided that the interface is smooth at (x_i^*, y_j^*). We need to express all the quantities with the superscript $+$ in terms of those quantities with the superscript $-$.

First, using the same approach as that in [15] and [16], we easily get the following jump conditions for u_1 and u_2,

$$u_1^+(\chi(\eta), \eta)|_{\eta=0} = u_1^-(\chi(\eta), \eta)|_{\eta=0}, \tag{4.1}$$

$$u_2^+(\chi(\eta), \eta)|_{\eta=0} = u_2^-(\chi(\eta), \eta)|_{\eta=0}, \tag{4.2}$$

$$[u_{1\xi}]\chi'' + [u_{1\eta\eta}] = 0, \qquad [u_{2\xi}]\chi'' + [u_{2\eta\eta}] = 0. \tag{4.3}$$

The jump (4.3) can be rewritten as

$$u_{1\eta\eta}^+ = u_{1\eta\eta}^- + \chi'' u_{1\xi}^- - \chi'' u_{1\xi}^+, \tag{4.4}$$

$$u_{2\eta\eta}^+ = u_{2\eta\eta}^- + \chi'' u_{2\xi}^- - \chi'' u_{2\xi}^+. \tag{4.5}$$

Notice that, in the local coordinate system, the unit normal at any point $(\chi(\eta), \eta)$ on the interface is

$$\widehat{\mathbf{n}} = (\widehat{n_1}, \widehat{n_2})^T = \left(\frac{1}{\sqrt{1 + (\chi'(\eta))^2}}, \frac{-\chi'(\eta)}{\sqrt{1 + (\chi'(\eta))^2}} \right)^T. \tag{4.6}$$

Therefore, at (x_i^*, y_j^*), or $(0,0)$ in the local coordinate system, we have

$$\begin{aligned} \widehat{n_1}_\eta|_{\eta=0} &= -\left. \frac{\chi'(\eta)\chi''(\eta)}{(1 + (\chi'(\eta))^2)^{\frac{3}{2}}} \right|_{\eta=0}, \\ \widehat{n_2}_\eta|_{\eta=0} &= \left. \frac{-\chi''(\eta)}{\sqrt{1 + (\chi'(\eta))^2}} \right|_{\eta=0} + \left. \frac{(\chi'(\eta))^2 \chi''(\eta)}{(1 + (\chi'(\eta))^2)^{\frac{3}{2}}} \right|_{\eta=0}. \end{aligned}$$

Recalling that $\chi(0) = 0$ and $\chi'(0) = 0$, we have, at (x_i^*, y_j^*), or $(0,0)$ in the local coordinate system, that

$$\widehat{n_1} = 1, \qquad \widehat{n_2} = 0, \tag{4.7}$$

$$\widehat{n_1}_\eta = 0, \qquad \widehat{n_2}_\eta = -\chi''(0). \tag{4.8}$$

In the x-y coordinate system, we have $\mathbf{n} = (n_1, n_2)^T$. The relation between $(n_1, n_2)^T$ and $(\widehat{n_1}, \widehat{n_2})^T$ is determined by

$$\begin{cases} n_1 = \widehat{n_1} * \cos\theta - \widehat{n_2} * \sin\theta, \\ n_2 = \widehat{n_1} * \sin\theta + \widehat{n_2} * \cos\theta. \end{cases}$$

Hence, by (4.7) and (4.8), we obtain

$$n_1 = \cos\theta, \qquad n_2 = \sin\theta, \tag{4.9}$$

$$n_{1\eta}|_{\eta=0} = \widehat{n_1}_\eta|_{\eta=0}\cos\theta - \widehat{n_2}_\eta|_{\eta=0}\sin\theta = \chi''(0)\sin\theta, \tag{4.10}$$
$$n_{2\eta}|_{\eta=0} = \widehat{n_1}_\eta|_{\eta=0}\sin\theta + \widehat{n_2}_\eta|_{\eta=0}\cos\theta = -\chi''(0)\cos\theta. \tag{4.11}$$

These relations will be used to derive the interface relations for $u^+_{1\xi\eta}$ and $u^+_{2\xi\eta}$.

The following result is required to find the interface relations so every quantity from $u^+_{1\xi}$, $u^+_{2\xi}$, $u^+_{1\xi\eta}$, $u^+_{2\xi\eta}$, $u^+_{1\xi\xi}$ and $u^+_{2\xi\xi}$, can be expressed in terms of u^-_1, u^-_{1x}, u^-_{1y}, u^-_{1xx}, u^-_{1yy}, u^-_{1xy}, u^-_2, u^-_{2x}, u^-_{2y}, u^-_{2xx}, u^-_{2yy}, and u^-_{2xy}. Its proof is trivial, and is omitted.

Lemma 4.1. If $Det = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, then the system of linear equations

$$\begin{cases} a_{11}x + a_{12}y &= b_{11}z_1 + b_{12}z_2 + \cdots + b_{1n}z_n \\ a_{21}x + a_{22}y &= b_{21}z_1 + b_{22}z_2 + \cdots + b_{2n}z_n \end{cases}$$

has the unique solution

$$\begin{cases} x = \frac{1}{Det}\left(\begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} z_1 + \begin{vmatrix} b_{12} & a_{12} \\ b_{22} & a_{22} \end{vmatrix} z_2 + \cdots + \begin{vmatrix} b_{1n} & a_{12} \\ b_{2n} & a_{22} \end{vmatrix} z_n \right) \\ y = \frac{1}{Det}\left(\begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix} z_1 + \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} z_2 + \cdots + \begin{vmatrix} a_{11} & b_{1n} \\ a_{21} & b_{2n} \end{vmatrix} z_n \right) \end{cases}.$$

4.1 Interface relations for $u^+_{1\xi}$ and $u^+_{2\xi}$

Rewrite the interface conditions (2.4) and (2.5) as follow:

$$\begin{aligned} &n_1(\alpha^+ u^+_{1x} + \beta^+ u^+_{2y}) + n_2\mu^+(u^+_{1y} + u^+_{2x}) \\ &\quad = n_1(\alpha^- u^-_{1x} + \beta^- u^-_{2y}) + n_2\mu^-(u^-_{1y} + u^-_{2x}) + \phi(x,y), \\ &n_1\mu^+(u^+_{1y} + u^+_{2x}) + n_2(\beta^+ u^+_{1x} + \alpha^+ u^+_{2y}) \\ &\quad = n_1\mu^-(u^-_{1y} + u^-_{2x}) + n_2(\beta^- u^-_{1x} + \alpha^- u^-_{2y}) + \psi(x,y), \end{aligned}$$

where $\mathbf{n} = (n_1, n_2)^T = (n_1(\xi,\eta), n_2(\xi,\eta))^T$ is the unit normal to the interface and

$$\begin{aligned} \alpha^+ &= \frac{2\mu^+}{1-2\nu^+}(1-\nu^+), \quad \beta^+ = \frac{2\mu^+}{1-2\nu^+}\nu^+, \\ \alpha^- &= \frac{2\mu^-}{1-2\nu^-}(1-\nu^-), \quad \beta^- = \frac{2\mu^-}{1-2\nu^-}\nu^-. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= u_{1\xi}\xi_x + u_{1\eta}\eta_x = \cos(\theta)u_{1\xi} - \sin(\theta)u_{1\eta}, \\ \frac{\partial u_1}{\partial y} &= u_{1\xi}\xi_y + u_{1\eta}\eta_y = \sin(\theta)u_{1\xi} + \cos(\theta)u_{1\eta}, \\ \frac{\partial u_2}{\partial x} &= u_{2\xi}\xi_x + u_{2\eta}\eta_x = \cos(\theta)u_{2\xi} - \sin(\theta)u_{2\eta}, \\ \frac{\partial u_2}{\partial y} &= u_{2\xi}\xi_y + u_{2\eta}\eta_y = \sin(\theta)u_{2\xi} + \cos(\theta)u_{2\eta}. \end{aligned}$$

Thus, in the local ξ-η coordinate system, we have

$$(n_1\alpha^+\cos\theta + n_2\mu^+\sin\theta)u^+_{1\xi} + (-n_1\alpha^+\sin\theta + n_2\mu^+\cos\theta)u^+_{1\eta}$$

$$
\begin{aligned}
&+(n_1\beta^+\sin\theta+n_2\mu^+\cos\theta)u_{2\xi}^+ + (n_1\beta^+\cos\theta-n_2\mu^+\sin\theta)u_{2\eta}^+\\
&=(n_1\alpha^-\cos\theta+n_2\mu^-\sin\theta)u_{1\xi}^- + (-n_1\alpha^-\sin\theta+n_2\mu^-\cos\theta)u_{1\eta}^-\\
&+(n_1\beta^-\sin\theta+n_2\mu^-\cos\theta)u_{2\xi}^- + (n_1\beta^-\cos\theta-n_2\mu^-\sin\theta)u_{2\eta}^-\\
&+\phi(\xi,\eta), \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&(\mu^+n_1\sin\theta+n_2\beta^+\cos\theta)u_{1\xi}^+ + (n_1\mu^+\cos\theta+n_2\alpha^+\sin\theta)u_{2\xi}^+\\
&+(n_1\mu^+\cos\theta-n_2\beta^+\sin\theta)u_{1\eta}^+ + (-\mu^+n_1\sin\theta+n_2\alpha^+\cos\theta)u_{2\eta}^+\\
&=(\mu^-n_1\sin\theta+n_2\beta^-\cos\theta)u_{1\xi}^- + (n_1\mu^-\cos\theta+n_2\alpha^-\sin\theta)u_{2\xi}^-\\
&+(n_1\mu^-\cos\theta-n_2\beta^-\sin\theta)u_{1\eta}^- + (-\mu^-n_1\sin\theta+n_2\alpha^-\cos\theta)u_{2\eta}^-\\
&+\psi(\xi,\eta). \qquad (4.13)
\end{aligned}
$$

Using the fact that $n_1 = \cos\theta$, $n_2 = \sin\theta$ at $(0,0)$ in the local coordinate system and the jump conditions (4.1) and (4.2), we can solve for $u_{1\xi}^+$ and $u_{2\xi}^+$ from equations (4.12) and (4.13) in terms of u_1^-, $u_{1\xi}^-$, $\cdots$,$u_{2\xi\eta}^-$. In fact, using (4.9) we can rewrite the equations (4.12) and (4.13) as

$$
\begin{aligned}
&(\alpha^+\cos^2\theta+\mu^+\sin^2\theta)u_{1\xi}^+ + (\beta^++\mu^+)\sin\theta\cos\theta u_{2\xi}^+\\
&\quad=(\alpha^-\cos^2\theta+\mu^-\sin^2\theta)u_{1\xi}^- + (-\alpha^-+\mu^-)\sin\theta\cos\theta u_{1\eta}^-\\
&\quad-(-\alpha^++\mu^+)\sin\theta\cos\theta u_{1\eta}^+ - (\beta^+\cos^2\theta-\mu^+\sin^2\theta)u_{2\eta}^+\\
&\quad+(\beta^-+\mu^-)\sin\theta\cos\theta u_{2\xi}^- + (\beta^-\cos^2\theta-\mu^-\sin^2\theta)u_{2\eta}^-\\
&\quad+\phi(\xi,\eta), \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
&(\mu^++\beta^+)\sin\theta\cos\theta u_{1\xi}^+ + (\mu^+\cos^2\theta+\alpha^+\sin^2\theta)u_{2\xi}^+\\
&\quad=(\mu^-+\beta^-)\sin\theta\cos\theta u_{1\xi}^- + (\mu^-\cos^2\theta+\alpha^-\sin^2\theta)u_{2\xi}^-\\
&\quad-(\mu^+\cos^2\theta-\beta^+\sin^2\theta)u_{1\eta}^+ - (-\mu^++\alpha^+)\sin\theta\cos\theta u_{2\eta}^+\\
&\quad+(\mu^-\cos^2\theta-\beta^-\sin^2\theta)u_{1\eta}^- + (-\mu^-+\alpha^-)\sin\theta\cos\theta u_{2\eta}^-\\
&\quad+\psi(\xi,\eta). \qquad (4.15)
\end{aligned}
$$

Since $u_{1\eta}^+ = u_{1\eta}^-$ and $u_{2\eta}^+ = u_{2\eta}^-$, the right hand sides of (4.14) and (4.15) become

$$
\begin{aligned}
b_1 \;=\;& (\alpha^-\cos^2\theta+\mu^-\sin^2\theta)u_{1\xi}^-\\
&+[(\alpha^+-\alpha^-)-(\mu^+-\mu^-)]\sin\theta\cos\theta u_{1\eta}^- + (\beta^-+\mu^-)\sin\theta\cos\theta u_{2\xi}^-\\
&+[(\mu^+-\mu^-)\sin^2\theta-(\beta^+-\beta^-)\cos^2\theta]u_{2\eta}^- + \phi(\xi,\eta),\\
b_2 \;=\;& (\mu^-+\beta^-)\sin\theta\cos\theta u_{1\xi}^-\\
&+[(\beta^+-\beta^-)\sin^2\theta-(\mu^+-\mu^-)\cos^2\theta]u_{1\eta}^- + (\mu^-\cos^2\theta+\alpha^-\sin^2\theta)u_{2\xi}^-\\
&+[(\mu^+-\mu^-)-(\alpha^+-\alpha^-)]\sin\theta\cos\theta u_{2\eta}^- + \psi(\xi,\eta).
\end{aligned}
\qquad (4.16)
$$

Let

$$
\begin{aligned}
a_{11} &= \cos^2\theta\alpha^+ + \sin^2\theta\mu^+, & a_{12} &= (\beta^++\mu^+)\sin\theta\cos\theta,\\
a_{21} &= (\mu^++\beta^+)\sin\theta\cos\theta, & a_{22} &= \mu^+\cos^2\theta+\alpha^+\sin^2\theta.
\end{aligned}
$$

The determinant of the coefficient matrix for the linear system (4.14) and (4.15) is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \cos^2\theta\alpha^+ + \sin^2\theta\mu^+ & (\beta^+ + \mu^+)\sin\theta\cos\theta \\ (\mu^+ + \beta^+)\sin\theta\cos\theta & \mu^+\cos^2\theta + \alpha^+\sin^2\theta \end{vmatrix} = \alpha^+\mu^+.$$

Since

$$\alpha^+\mu^+ = \frac{2(\mu^+)^2}{1-2\nu^+}(1-\nu^+) \neq 0, \tag{4.17}$$

the system (4.14) and (4.15) has the unique solution

$$u^+_{1\xi} = \frac{1}{\alpha^+\mu^+}\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \frac{1}{\alpha^+\mu^+}(a_{22}b_1 - a_{12}b_2), \tag{4.18}$$

$$u^+_{2\xi} = \frac{1}{\alpha^+\mu^+}\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \frac{1}{\alpha^+\mu^+}(a_{11}b_2 - a_{21}b_1. \tag{4.19}$$

Consequently, the values $u^+_{1\xi}$ and $u^+_{2\xi}$ can be expressed in terms of u^-_1, $u^-_{1\xi}$, $\cdots$, $u^-_{2\xi\eta}$. Moreover, plugging (4.18) and (4.19) into (4.4) and (4.5), we can also express $u^+_{1\eta\eta}$ and $u^+_{2\eta\eta}$ in terms of those values in the $-$ side.

4.2 Interface relations for $u^+_{1\xi\eta}$ and $u^+_{2\xi\eta}$

To find the interface relations for $u^+_{1\xi\eta}$ and $u^+_{2\xi\eta}$, we differentiate (4.12) and (4.13) with respect to η. After some calculations, we obtain the following system of linear equations about $u^+_{1\xi\eta}$ and $u^+_{2\xi\eta}$

$$\begin{cases} A_{11}u^+_{1\xi\eta} + A_{12}u^+_{2\xi\eta} = B_1, \\ A_{21}u^+_{1\xi\eta} + A_{22}u^+_{2\xi\eta} = B_2, \end{cases} \tag{4.20}$$

where the coefficients $A_{11}, A_{12}, A_{21}, A_{22}$ are given by

$$A_{11} = n_1\alpha^+\cos\theta + n_2\mu^+\sin\theta, \qquad A_{12} = n_1\beta^+\sin\theta + n_2\mu^+\cos\theta,$$
$$A_{21} = n_1\mu^+\sin\theta + n_2\beta^+\cos\theta, \qquad A_{22} = n_1\mu^+\cos\theta + n_2\alpha^+\sin\theta.$$

The terms B_1 and B_2 involve $u^+_{1\xi}$, $u^+_{2\xi}$, $u^+_{1\eta\eta}$, $u^+_{2\eta\eta}$, and u^-_1, $u^-_{1\xi}$, $\cdots$, $u^-_{2\xi\eta}$. However, since we have already obtained that $u^+_{1\xi}$, $u^+_{2\xi}$, $u^+_{1\eta\eta}$, and $u^+_{2\eta\eta}$ can be expressed as a linear combination of u^-_1, $u^-_{1\xi}$, $\cdots$, $u^-_{2\xi\eta}$, the values B_1 and B_2 can also be expressed in terms of these quantities as well. So we set

$$\begin{aligned} B_1 \;&=\; b^1_1u^-_1 + b^1_2u^-_{1\xi} + b^1_3u^-_{1\eta} + b^1_4u^-_{1\xi\xi} + b^1_5u^-_{1\eta\eta} + b^1_6u^-_{1\xi\eta} \\ &\quad + b^1_7u^-_2 + b^1_8u^-_{2\xi} + b^1_9u^-_{2\eta} + b^1_{10}u^-_{2\xi\xi} + b^1_{11}u^-_{2\eta\eta} + b^1_{12}u^-_{2\xi\eta} + b^1_0, \\ B_2 \;&=\; b^2_1u^-_1 + b^2_2u^-_{1\xi} + b^2_3u^-_{1\eta} + b^2_4u^-_{1\xi\xi} + b^2_5u^-_{1\eta\eta} + b^2_6u^-_{1\xi\eta} \\ &\quad + b^2_7u^-_2 + b^2_8u^-_{2\xi} + b^2_9u^-_{2\eta} + b^2_{10}u^-_{2\xi\xi} + b^2_{11}u^-_{2\eta\eta} + b^2_{12}u^-_{2\xi\eta} + b^2_0. \end{aligned}$$

Since $n_1 = \cos\theta$ and $n_2 = \sin\theta$, the determinant of the coefficient matrix of the system (4.20) is

$$DET = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} \cos^2\theta\alpha^+ + \sin^2\theta\mu^+ & (\beta^+ + \mu^+)\sin\theta\cos\theta \\ (\mu^+ + \beta^+)\sin\theta\cos\theta & \mu^+\cos^2\theta + \alpha^+\sin^2\theta \end{vmatrix} = \alpha^+\mu^+ \neq 0.$$

Thus, from Lemma 4.1, the solution to (4.20) is

$$
\begin{aligned}
u^+_{1\xi\eta} &= \frac{1}{DET}\begin{vmatrix} b^1_1 & A_{12} \\ b^2_1 & A_{22}\end{vmatrix} u^-_1 + \frac{1}{DET}\begin{vmatrix} b^1_2 & A_{12} \\ b^2_2 & A_{22}\end{vmatrix} u^-_{1\xi} + \frac{1}{DET}\begin{vmatrix} b^1_3 & A_{12} \\ b^2_3 & A_{22}\end{vmatrix} u^-_{1\eta} \\
&+\frac{1}{DET}\begin{vmatrix} b^1_4 & A_{12} \\ b^2_4 & A_{22}\end{vmatrix} u^-_{1\xi\xi} + \frac{1}{DET}\begin{vmatrix} b^1_5 & A_{12} \\ b^2_5 & A_{22}\end{vmatrix} u^-_{1\eta\eta} + \frac{1}{DET}\begin{vmatrix} b^1_6 & A_{12} \\ b^2_6 & A_{22}\end{vmatrix} u^-_{1\xi\eta} \\
&+\frac{1}{DET}\begin{vmatrix} b^1_7 & A_{12} \\ b^2_7 & A_{22}\end{vmatrix} u^-_2 + \frac{1}{DET}\begin{vmatrix} b^1_8 & A_{12} \\ b^2_8 & A_{22}\end{vmatrix} u^-_{2\xi} + \frac{1}{DET}\begin{vmatrix} b^1_9 & A_{12} \\ b^2_9 & A_{22}\end{vmatrix} u^-_{2\eta} \\
&+\frac{1}{DET}\begin{vmatrix} b^1_{10} & A_{12} \\ b^2_{10} & A_{22}\end{vmatrix} u^-_{2\xi\xi} + \frac{1}{DET}\begin{vmatrix} b^1_{11} & A_{12} \\ b^2_{11} & A_{22}\end{vmatrix} u^-_{2\eta\eta} \\
&+\frac{1}{DET}\begin{vmatrix} b^1_{12} & A_{12} \\ b^2_{12} & A_{22}\end{vmatrix} u^-_{2\xi\eta} + \frac{1}{DET}\begin{vmatrix} b^1_0 & A_{12} \\ b^2_0 & A_{22}\end{vmatrix}, \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
u^+_{2\xi\eta} &= \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_1 \\ A_{21} & b^2_1\end{vmatrix} u^-_1 + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_2 \\ A_{21} & b^2_2\end{vmatrix} u^-_{1\xi} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_3 \\ A_{21} & b^2_3\end{vmatrix} u^-_{1\eta} \\
&+\frac{1}{DET}\begin{vmatrix} A_{11} & b^1_4 \\ A_{21} & b^2_4\end{vmatrix} u^-_{1\xi\xi} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_5 \\ A_{21} & b^2_5\end{vmatrix} u^-_{1\eta\eta} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_6 \\ A_{21} & b^2_6\end{vmatrix} u^-_{1\xi\eta} \\
&+\frac{1}{DET}\begin{vmatrix} A_{11} & b^1_7 \\ A_{21} & b^2_7\end{vmatrix} u^-_2 + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_8 \\ A_{21} & b^2_8\end{vmatrix} u^-_{2\xi} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_9 \\ A_{21} & b^2_9\end{vmatrix} u^-_{2\eta} \\
&+\frac{1}{DET}\begin{vmatrix} A_{11} & b^1_{10} \\ A_{21} & b^2_{10}\end{vmatrix} u^-_{2\xi\xi} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_{11} \\ A_{21} & b^2_{11}\end{vmatrix} u^-_{2\eta\eta} \\
&+\frac{1}{DET}\begin{vmatrix} A_{11} & b^1_{12} \\ A_{21} & b^2_{12}\end{vmatrix} u^-_{2\xi\eta} + \frac{1}{DET}\begin{vmatrix} A_{11} & b^1_0 \\ A_{21} & b^2_0\end{vmatrix}. \qquad (4.22)
\end{aligned}
$$

These are the desired relations.

4.3 Interface relations for $u^+_{1\xi\xi}$ and $u^+_{2\xi\xi}$

From the differential equations (3.2) and (3.3), we have

$$
\begin{aligned}
&(\cos^2\theta + 1 - 2\nu^+)u^+_{1\xi\xi} - 2\sin\theta\cos\theta u^+_{1\xi\eta} + (\sin^2\theta + 1 - 2\nu^+)u^+_{1\eta\eta} \\
&\quad + \sin\theta\cos\theta u^+_{2\xi\xi} + (\cos^2\theta - \sin^2\theta)u^+_{2\xi\eta} - \sin\theta\cos\theta u^+_{2\eta\eta} - f^+ \\
= \;&(\cos^2\theta + 1 - 2\nu^-)u^-_{1\xi\xi} - 2\sin\theta\cos\theta u^-_{1\xi\eta} + (\sin^2\theta + 1 - 2\nu^-)u^-_{1\eta\eta} \\
&+ \sin\theta\cos\theta u^-_{2\xi\xi} + (\cos^2\theta - \sin^2\theta)u^-_{2\xi\eta} - \sin\theta\cos\theta u^-_{2\eta\eta} - f^-, \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
&(\sin^2\theta + 1 - 2\nu^+)u^+_{2\xi\xi} + 2\sin\theta\cos\theta u^+_{2\xi\eta} + (\cos^2\theta + 1 - 2\nu^+)u^+_{2\eta\eta} \\
&\quad + \sin\theta\cos\theta u^+_{1\xi\xi} + (\cos^2\theta - \sin^2\theta)u^+_{1\xi\eta} - \sin\theta\cos\theta u^+_{1\eta\eta} - g^+ \\
= \;&(\sin^2\theta + 1 - 2\nu^-)u^-_{2\xi\xi} + 2\sin\theta\cos\theta u^-_{2\xi\eta} + (\cos^2\theta + 1 - 2\nu^-)u^-_{2\eta\eta} \\
&+ \sin\theta\cos\theta u^-_{1\xi\xi} + (\cos^2\theta - \sin^2\theta)u^-_{1\xi\eta} - \sin\theta\cos\theta u^-_{1\eta\eta} - g^-. \qquad (4.24)
\end{aligned}
$$

We need to solve the above equations for $u^+_{1\xi\xi}$ and $u^+_{1\xi\xi}$. Observe that the determinant of the coefficient matrix of the above system is

$$
\begin{vmatrix} 1 - 2\nu^+ + \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & 1 - 2\nu^+ + \sin^2\theta \end{vmatrix} = 2(1-2\nu^+)(1-\nu^+) \neq 0,
$$

By (4.4), (4.5), (4.21), and (4.22), we can solve (4.23) and (4.24) for $u^+_{1\xi\xi}$ and $u^+_{1\xi\xi}$, which are expressed in terms of u^-_1, $u^-_{1\xi}$, $\cdots$, $u^-_{2\xi\eta}$.

5 Derivation of the Finite Difference Scheme at Type I Irregular Grid Points

We are now ready to derive the finite difference scheme for irregular grid points of type I. We first rewrite the finite difference scheme (2.1) as follows:

$$\left\{ \begin{array}{l} \sum_{k=1}^{9} \alpha_k (U_1)_{i+i_k,j+j_k} + \sum_{k=1}^{9} \beta_k (U_2)_{i+i_k,j+j_k} = f(x_i,y_j) + C^1_{ij}, \\ \sum_{k=1}^{9} \gamma_k (U_1)_{i+i_k,j+j_k} + \sum_{k=1}^{9} \delta_k (U_2)_{i+i_k,j+j_k} = g(x_i,y_j) + C^2_{ij}. \end{array} \right. \tag{5.1}$$

We use the undetermined coefficient method to determine all the coefficients α_k, β_k, γ_k, and δ_k.

Denote the ξ-η coordinates of the nine grid points in the finite difference stencil (see Figure 5.1.)

$$(x_{i-1},y_{j-1}), (x_{i-1},y_j), (x_{i-1},y_{j+1}), (x_i,y_{j-1}), (x_i,y_j), \\ (x_i,y_{j+1}), (x_{i+1},y_{j-1}), (x_{i+1},y_j), (x_{i+1},y_{j+1})$$

as

$$(\xi_1,\eta_1), (\xi_2,\eta_2), (\xi_3,\eta_3), (\xi_4,\eta_4), (\xi_5,\eta_5), (\xi_6,\eta_6), (\xi_7,\eta_7), (\xi_8,\eta_8), (\xi_9,\eta_9).$$

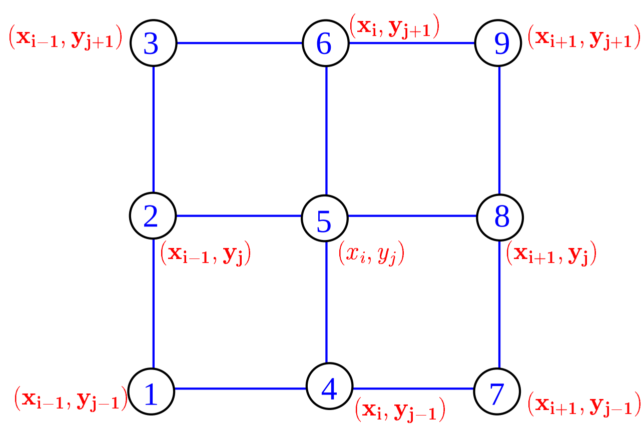

Figure 5.1: The labels of the nine grid points.

The local truncation errors at a grid point (x_i, y_j) are defined as

$$\begin{aligned} T^1_{ij} &= \sum_{k=1}^{9} \alpha_k u_1(x_{i+i_k}, y_{j+j_k}) + \sum_{k=1}^{9} \beta_k u_2(x_{i+i_k}, y_{j+j_k}) - f(x_i,y_j) - C^1_{ij} \\ &= \sum_{k=1}^{9} \alpha_k u_1(\xi_k,\eta_k) + \sum_{k=1}^{9} \beta_k u_2(\xi_k,\eta_k) - f(\xi_i,\eta_j) - C^1_{ij}, \end{aligned} \tag{5.2}$$

$$
\begin{aligned}
T^2_{ij} &= \sum_{k=1}^{9} \gamma_k u_1(x_{i+i_k}, y_{j+j_k}) + \sum_{k=1}^{9} \delta_k u_2(x_{i+i_k}, y_{j+j_k}) - g(x_i, y_j) - C^2_{ij} \\
&= \sum_{k=1}^{9} \gamma_k u_1(\xi_k, \eta_k) + \sum_{k=1}^{9} \delta_k u_2(\xi_k, \eta_k) - g(\xi_i, \eta_j) - C^2_{ij}.
\end{aligned} \tag{5.3}
$$

We expand $u_1(\xi_k, \eta_k)$ and $u_2(\xi_k, \eta_k)$ about $(0, 0)$ in Taylor series in the local coordinate system from each side of the interface to get

$$
\begin{aligned}
u_1(\xi_k, \eta_k) &= u_1^{\pm} + \xi_k u^{\pm}_{1\xi} + \eta_k u^{\pm}_{1\eta} + \frac{1}{2}\xi_k^2 u^{\pm}_{1\xi\xi} + \xi_k \eta_k u^{\pm}_{1\xi\eta} + \frac{1}{2}\eta_k^2 u^{\pm}_{1\eta\eta} \\
&\quad + \mathcal{O}(h^3), \qquad (5.4) \\
u_2(\xi_k, \eta_k) &= u_2^{\pm} + \xi_k u^{\pm}_{2\xi} + \eta_k u^{\pm}_{2\eta} + \frac{1}{2}\xi_k^2 u^{\pm}_{2\xi\xi} + \xi_k \eta_k u^{\pm}_{2\xi\eta} + \frac{1}{2}\eta_k^2 u^{\pm}_{2\eta\eta} \\
&\quad + \mathcal{O}(h^3), \qquad (5.5)
\end{aligned}
$$

where the $+$ or $-$ is chosen depending on whether (ξ_k, η_k) lies on the $+$ or $-$ side of Γ. Carrying out such expansion for each point involved in (5.2) and (5.3), we obtain the following expressions of T^1_{ij} and T^2_{ij} for the local truncation errors as linear combinations of the values $u_1^{\pm}$, $u^{\pm}_{1\xi}$, $u^{\pm}_{1\eta}$, $u^{\pm}_{1\xi\xi}$, $u^{\pm}_{1\xi\eta}$, $u^{\pm}_{1\eta\eta}$, $u_2^{\pm}$, $u^{\pm}_{2\xi}$, $u^{\pm}_{2\eta}$, $u^{\pm}_{2\xi\xi}$, $u^{\pm}_{2\xi\eta}$, and $u^{\pm}_{2\eta\eta}$:

$$
\begin{aligned}
T^1_{ij} &= a_1 u_1^- + a_2 u_1^+ + a_3 u^-_{1\xi} + a_4 u^+_{1\xi} + a_5 u^-_{1\eta} + a_6 u^+_{1\eta} \\
&\quad + a_7 u^-_{1\xi\xi} + a_8 u^+_{1\xi\xi} + a_9 u^-_{1\eta\eta} + a_{10} u^+_{1\eta\eta} + a_{11} u^-_{1\xi\eta} + a_{12} u^+_{1\xi\eta} \\
&\quad + b_1 u_2^- + b_2 u_2^+ + b_3 u^-_{2\xi} + b_4 u^+_{2\xi} + b_5 u^-_{2\eta} + b_6 u^+_{2\eta} \qquad (5.6) \\
&\quad + b_7 u^-_{1\xi\xi} + b_8 u^+_{1\xi\xi} + b_9 u^-_{1\eta\eta} + b_{10} u^+_{1\eta\eta} + b_{11} u^-_{1\xi\eta} + b_{12} u^+_{1\xi\eta} \\
&\quad - f^-_{ij} - C^1_{ij} + \mathcal{O}(h), \\
T^2_{ij} &= c_1 u_1^- + c_2 u_1^+ + c_3 u^-_{1\xi} + c_4 u^+_{1\xi} + c_5 u^-_{1\eta} + c_6 u^+_{1\eta} \\
&\quad + c_7 u^-_{1\xi\xi} + c_8 u^+_{1\xi\xi} + c_9 u^-_{1\eta\eta} + c_{10} u^+_{1\eta\eta} + c_{11} u^-_{1\xi\eta} + c_{12} u^+_{1\xi\eta} \\
&\quad + d_1 u_2^- + b_2 u_2^+ + d_3 u^-_{2\xi} + d_4 u^+_{2\xi} + d_5 u^-_{2\eta} + d_6 u^+_{2\eta} \qquad (5.7) \\
&\quad + d_7 u^-_{2\xi\xi} + d_8 u^+_{2\xi\xi} + d_9 u^-_{2\eta\eta} + d_{10} u^+_{2\eta\eta} + d_{11} u^-_{2\xi\eta} + d_{12} u^+_{2\xi\eta} \\
&\quad - g^-_{ij} - C^2_{ij} + \mathcal{O}(h).
\end{aligned}
$$

All the coefficients a_k, b_k, c_k, and d_k depend only on the position of the stencil relative to the interface. In particular, they are independent of the functions u_1, u_2, f, and g. If we define the index sets K^+ and K^- by

$$
K^{\pm} = \{k : (\xi_k, \eta_k) \text{ is on the } \pm \text{ side of } \Gamma\},
$$

then the coefficients a_k, b_k, c_k, and d_k are given by

$$
\begin{array}{lll}
a_1 = \sum_{k \in K^-} \alpha_k, & a_2 = \sum_{k \in K^+} \alpha_k, & a_3 = \sum_{k \in K^-} \xi_k \alpha_k, \\
a_4 = \sum_{k \in K^+} \xi_k \alpha_k, & a_5 = \sum_{k \in K^-} \eta_k \alpha_k, & a_6 = \sum_{k \in K^+} \eta_k \alpha_k, \\
a_7 = \sum_{k \in K^-} \frac{1}{2}\xi_k^2 \alpha_k, & a_8 = \sum_{k \in K^+} \frac{1}{2}\xi_k^2 \alpha_k, & a_9 = \sum_{k \in K^-} \frac{1}{2}\eta_k^2 \alpha_k, \\
a_{10} = \sum_{k \in K^+} \frac{1}{2}\eta_k^2 \alpha_k, & a_{11} = \sum_{k \in K^-} \xi_k \eta_k \alpha_k, & a_{12} = \sum_{k \in K^+} \xi_k \eta_k \alpha_k,
\end{array} \tag{5.8}
$$

$$
\begin{array}{lll}
b_1 = \sum_{m\in K^-} \beta_m, & b_2 = \sum_{m\in K^+} \beta_m, & b_3 = \sum_{m\in K^-} \xi_m\beta_m, \\
b_4 = \sum_{m\in K^+} \xi_m\beta_m, & b_5 = \sum_{m\in K^-} \eta_m\beta_m, & b_6 = \sum_{m\in K^+} \eta_m\beta_m, \\
b_7 = \sum_{m\in K^-} \frac{1}{2}\xi_m^2\beta_m, & b_8 = \sum_{m\in K^+} \frac{1}{2}\xi_m^2\beta_m, & b_9 = \sum_{m\in K^-} \frac{1}{2}\eta_m^2\beta_m, \\
b_{10} = \sum_{m\in K^+} \frac{1}{2}\eta_m^2\beta_m, & b_{11} = \sum_{m\in K^-} \xi_m\eta_m\beta_m, & b_{12} = \sum_{m\in K^+} \xi_m\eta_m\beta_m,
\end{array}
\tag{5.9}
$$

$$
\begin{array}{lll}
c_1 = \sum_{k\in K^-} \gamma_k, & c_2 = \sum_{k\in K^+} \gamma_k, & c_3 = \sum_{k\in K^-} \xi_k\gamma_k, \\
c_4 = \sum_{k\in K^+} \xi_k\gamma_k, & c_5 = \sum_{k\in K^-} \eta_k\gamma_k, & c_6 = \sum_{k\in K^+} \eta_k\gamma_k, \\
c_7 = \sum_{k\in K^-} \frac{1}{2}\xi_k^2\gamma_k, & c_8 = \sum_{k\in K^+} \frac{1}{2}\xi_k^2\gamma_k, & c_9 = \sum_{k\in K^-} \frac{1}{2}\eta_k^2\gamma_k, \\
c_{10} = \sum_{k\in K^+} \frac{1}{2}\eta_k^2\gamma_k, & c_{11} = \sum_{k\in K^-} \xi_k\eta_k\gamma_k, & c_{12} = \sum_{k\in K^+} \xi_k\eta_k\gamma_k,
\end{array}
\tag{5.10}
$$

$$
\begin{array}{lll}
d_1 = \sum_{m\in K^-} \delta_m, & d_2 = \sum_{m\in K^+} \delta_m, & d_3 = \sum_{m\in K^-} \xi_m\delta_m, \\
d_4 = \sum_{m\in K^+} \xi_m\delta_m, & d_5 = \sum_{m\in K^-} \eta_m\delta_m, & d_6 = \sum_{m\in K^+} \eta_m\delta_m, \\
d_7 = \sum_{m\in K^-} \frac{1}{2}\xi_m^2\delta_m, & d_8 = \sum_{m\in K^+} \frac{1}{2}\xi_m^2\delta_m, & d_9 = \sum_{m\in K^-} \frac{1}{2}\eta_m^2\delta_m, \\
d_{10} = \sum_{m\in K^+} \frac{1}{2}\eta_m^2\delta_m, & d_{11} = \sum_{m\in K^-} \xi_m\eta_m\delta_m, & d_{12} = \sum_{m\in K^+} \xi_m\eta_m\delta_m.
\end{array}
\tag{5.11}
$$

Let $\overrightarrow{U_{1x}}$, $\overrightarrow{U_{1xx}}$, $\overrightarrow{U_{1xy}}$, $\overrightarrow{U_{2x}}$, $\overrightarrow{U_{2xx}}$, $\overrightarrow{U_{2xy}}$ be 12 dimensional column vectors with every coordinate for each vector corresponding one of the coefficients of u_1^-, $u_{1\xi}^-$, $u_{1\eta}^-$, $u_{1\xi\xi}^-$, $u_{1\eta\eta}^-$, $u_{1\xi\eta}^-$, u_2^-, $u_{2\xi}^-$, $u_{2\eta}^-$, $u_{2\xi\xi}^-$, $u_{2\eta\eta}^-$, $u_{2\xi\eta}^-$ in the expressions of $u_{1\xi}^+, u_{2\xi}^+, u_{1\xi\eta}^+, u_{2\xi\eta}^+, u_{1\xi\xi}^+, u_{2\xi\xi}^+$. We can then rewrite $u_{1\xi}^+, u_{2\xi}^+, u_{1\xi\eta}^+, u_{2\xi\eta}^+, u_{1\xi\xi}^+, u_{2\xi\xi}^+$ in terms of $\overrightarrow{U_{1x}}$, $\overrightarrow{U_{1xx}}$, $\overrightarrow{U_{1xy}}$, $\overrightarrow{U_{2x}}$, $\overrightarrow{U_{2xx}}$, $\overrightarrow{U_{2xy}}$. Let

$$
\overrightarrow{Vec_-} = \left[u_1^-, u_{1\xi}^-, u_{1\eta}^-, u_{1\xi\xi}^-, u_{1\eta\eta}^-, u_{1\xi\eta}^-, u_2^-, u_{2\xi}^-, u_{2\eta}^-, u_{2\xi\xi}^-, u_{2\eta\eta}^-, u_{2\xi\eta}^-\right] \tag{5.12}
$$

From the interface conditions we derived in Section 4, we have the following simple expression in the dot product form

$$
\left\{
\begin{array}{lcl}
u_{1\xi}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{1x}} + w_{1\xi_0}, \\
u_{2\xi}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{2x}} + w_{2\xi_0}, \\
u_{1\xi\xi}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{1xx}} + w_{1\xi\xi_0}, \\
u_{2\xi\xi}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{2xx}} + w_{2\xi\xi_0}, \\
u_{1\xi\eta}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{1xy}} + w_{1\xi\eta_0}, \\
u_{2\xi\eta}^+ & = & \overrightarrow{Vec_-}\cdot\overrightarrow{U_{2xy}} + w_{2\xi\eta_0}.
\end{array}
\right.
\tag{5.13}
$$

Using the jump conditions for $u_{1\eta\eta}^+$ and $u_{2\eta\eta}^+$ in (4.4) and (4.5), we can rewrite the truncation errors (5.6) and (5.7) as

$$
T_{ij}^1 = (a_1 + a_2)u_1^- + (a_3 + \chi'' a_{10})u_{1\xi}^- + (a_4 - \chi'' a_{10})u_{1\xi}^+
$$

$$
\begin{aligned}
&+(a_5+a_6)u^-_{1\eta}+[a_7-(\cos^2\theta+1-2\nu^-)]u^-_{1\xi\xi}+a_8u^+_{1\xi\xi}\\
&+[a_9+a_{10}-(\sin^2\theta+1-2\nu^-)]u^-_{1\eta\eta}\\
&+[a_{11}+2\sin\theta\cos\theta]u^-_{1\xi\eta}+a_{12}u^+_{1\xi\eta}\\
&+(b_1+b_2)u^-_2+(b_3+\chi''b_{10})u^-_{2\xi}+(b_4-\chi''b_{10})u^+_{2\xi}\\
&+(b_5+b_6)u^-_{2\eta}+(b_7-\sin\theta\cos\theta)u^-_{2\xi\xi}+b_8u^+_{2\xi\xi}\\
&+(b_9+b_{10}+\sin\theta\cos\theta)u^-_{2\eta\eta}+[b_{11}-(\cos^2\theta-\sin^2\theta)]u^-_{2\xi\eta}+b_{12}u^+_{2\xi\eta}\\
&+[(\cos^2\theta+1-2\nu^-)u^-_{1\xi\xi}-2\sin\theta\cos\theta u^-_{1\xi\eta}+(\sin^2\theta+1-2\nu^-)u^-_{1\eta\eta}\\
&+\sin\theta\cos\theta u^-_{2\xi\xi}+(\cos^2\theta-\sin^2\theta)u^-_{2\xi\eta}-\sin\theta\cos\theta u^-_{2\eta\eta}-f^-_{ij}]\\
&-C^1_{ij}+\mathcal{O}(h),
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
T^2_{ij}=\;&(c_1+c_2)u^-_1+(c_3+\chi''c_{10})u^-_{1\xi}+(c_4-\chi''c_{10})u^+_{1\xi}\\
&+(c_5+c_6)u^-_{1\eta}+(c_7-\sin\theta\cos\theta)u^-_{1\xi\xi}+c_8u^+_{1\xi\xi}\\
&+(c_9+c_{10}+\sin\theta\cos\theta)u^-_{1\eta\eta}\\
&+[c_{11}-(\cos^2\theta-\sin^2\theta)]u^-_{1\xi\eta}+c_{12}u^+_{1\xi\eta}\\
&+(d_1+d_2)u^-_2+(d_3+\chi''d_{10})u^-_{2\xi}+(d_4-\chi''d_{10})u^+_{2\xi}\\
&+(d_5+d_6)u^-_{2\eta}+[d_7-(\sin^2\theta+1-2\nu^-)]u^-_{2\xi\xi}+d_8u^+_{2\xi\xi}\\
&+[d_9+d_{10}-(\cos^2\theta+1-2\nu^-)]u^-_{2\eta\eta}\\
&+(d_{11}-2\sin\theta\cos\theta)u^-_{2\xi\eta}+d_{12}u^+_{2\xi\eta}\\
&+[(\sin^2\theta+1-2\nu^-)u^-_{2\xi\xi}+2\sin\theta\cos\theta u^-_{2\xi\eta}+(\cos^2\theta+1-2\nu^-)u^-_{2\eta\eta}\\
&+\sin\theta\cos\theta u^-_{1\xi\xi}+(\cos^2\theta-\sin^2\theta)u^-_{1\xi\eta}-\sin\theta\cos\theta u^-_{1\eta\eta}-g^-_{ij}]\\
&-C^1_{ij}+\mathcal{O}(h).
\end{aligned} \tag{5.15}
$$

Notice that from the equations (3.2) and (3.3), we have

$$
\begin{aligned}
&(\cos^2\theta+1-2\nu^-)u^-_{1\xi\xi}-2\sin\theta\cos\theta u^-_{1\xi\eta}+(\sin^2\theta+1-2\nu^-)u^-_{1\eta\eta}\\
&\quad+\sin\theta\cos\theta u^-_{2\xi\xi}+(\cos^2\theta-\sin^2\theta)u^-_{2\xi\eta}-\sin\theta\cos\theta u^-_{2\eta\eta}=f^-_{ij},\\
&(\sin^2\theta+1-2\nu^-)u^-_{2\xi\xi}+2\sin\theta\cos\theta u^-_{2\xi\eta}+(\cos^2\theta+1-2\nu^-)u^-_{2\eta\eta}\\
&\quad+\sin\theta\cos\theta u^-_{1\xi\xi}+(\cos^2\theta-\sin^2\theta)u^-_{1\xi\eta}-\sin\theta\cos\theta u^-_{1\eta\eta}=g^-_{ij}.
\end{aligned}
$$

From (5.12) and (5.13), we can rewrite the truncation errors T^1_{ij}, T^2_{ij} in (5.14) and (5.15) as

$$
\begin{aligned}
T^1_{ij}=\overrightarrow{Vec_-}\cdot\Big[&\overrightarrow{V_1}+(a_4-\chi''a_{10})\overrightarrow{U_{1x}}+a_8\overrightarrow{U_{1xx}}+a_{12}\overrightarrow{U_{1xy}}\\
&+(b_4-\chi''b_{10})\overrightarrow{U_{2x}}+b_8\overrightarrow{U_{2xx}}+b_{12}\overrightarrow{U_{2xy}}\Big]+\Delta_1+\mathcal{O}(h),
\end{aligned} \tag{5.16}
$$

$$
\begin{aligned}
T^2_{ij}=\overrightarrow{Vec_-}\cdot\Big[&\overrightarrow{V_2}+(c_4-\chi''c_{10})\overrightarrow{U_{1x}}+c_8\overrightarrow{U_{1xx}}+c_{12}\overrightarrow{U_{1xy}}\\
&+(d_4-\chi''d_{10})\overrightarrow{U_{2x}}+d_8\overrightarrow{U_{2xx}}+d_{12}\overrightarrow{U_{2xy}}\Big]+\Delta_2+\mathcal{O}(h),
\end{aligned} \tag{5.17}
$$

where $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ are twelve dimensional vectors similarly defined as $\overrightarrow{U_{1x}}$, $\overrightarrow{U_{1xx}}$, $\overrightarrow{U_{1xy}}$,

$\overrightarrow{U_{2x}}$, $\overrightarrow{U_{2xx}}$, $\overrightarrow{U_{2xy}}$, and

$$
\begin{array}{ll}
V_1(1) = a_1 + a_2, & V_1(2) = a_3 + \chi'' a_{10}, \\
V_1(3) = a_5 + a_6, & V_1(4) = a_7 - (\cos^2\theta + 1 - 2\nu^-) \\
V_1(5) = a_9 + a_{10} - (\sin^2\theta + 1 - 2\nu^-), & V_1(6) = a_{11} + 2\sin\theta\cos\theta, \\
V_1(7) = b_1 + b_2, & V_1(8) = b_3 + \chi'' b_{10}, \\
V_1(9) = b_5 + b_6, & V_1(10) = b_7 - \sin\theta\cos\theta, \\
V_1(11) = b_9 + b_{10} + \sin\theta\cos\theta, & V_1(12) = b_{11} - (\cos^2\theta - \sin^2\theta), \\
V_2(1) = c_1 + c_2, & V_2(2) = c_3 + \chi'' c_{10}, \\
V_2(3) = c_5 + c_6, & V_2(4) = c_7 - \sin\theta\cos\theta, \\
V_2(5) = c_9 + c_{10} + \sin\theta\cos\theta, & V_2(6) = c_{11} - (\cos^2\theta - \sin^2\theta), \\
V_2(7) = d_1 + d_2, & V_2(8) = d_3 + \chi'' d_{10}, \\
V_2(9) = d_5 + d_6, & V_2(10) = d_7 - (\sin^2\theta + 1 - 2\nu^-) \\
V_2(11) = d_9 + d_{10} - (\cos^2\theta + 1 - 2\nu^-), & V_2(12) = d_{11} - 2\sin\theta\cos\theta,
\end{array}
$$

$$
\begin{aligned}
\Delta_1 = {} & (a_4 - a_{10}\cdot\chi''(0))w_{1\xi_0} + a_8 w_{1\xi\xi_0} + a_{12} w_{1\xi\eta_0} \\
& +(b_4 - b_{10}\cdot\chi''(0))w_{2\xi_0} + b_8 w_{2\xi\xi_0} + b_{12} w_{2\xi\eta_0} - C^1_{i,j},
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
\Delta_2 = {} & (c_4 - c_{10}\cdot\chi''(0))w_{1\xi_0} + c_8 w_{1\xi\xi_0} + c_{12} w_{1\xi\eta_0} \\
& +(d_4 - d_{10}\cdot\chi''(0))w_{2\xi_0} + d_8 w_{2\xi\xi_0} + d_{12} w_{2\xi\eta_0} - C^2_{i,j},
\end{aligned} \tag{5.19}
$$

where $V1(i)$, $V2(i)$ are the i-th components of $\overrightarrow{V1}$ and $\overrightarrow{V2}$ respectively.

To minimize the truncation errors, we choose the coefficients α_k, β_k, γ_k, and δ_k so that the coefficients of u_1^-, $u_{1\xi}^-$, $u_{1\eta}^-$, $u_{1\xi\xi}^-$, $u_{1\eta\eta}^-$, $u_{1\xi\eta}^-$, u_2^-, $u_{2\xi}^-$, $u_{2\eta}^-$, $u_{2\xi\xi}^-$, $u_{2\eta\eta}^-$, and $u_{2\xi\eta}^-$ vanish in (5.16) and (5.17). Hence we set the following system of equations by (5.16) and (5.17):

$$
\begin{aligned}
& \overrightarrow{V_1} + (a_4 - \chi'' a_{10})\overrightarrow{U_{1x}} + a_8\overrightarrow{U_{1xx}} + a_{12}\overrightarrow{U_{1xy}} \\
& \qquad +(b_4 - \chi'' b_{10})\overrightarrow{U_{2x}} + b_8\overrightarrow{U_{2xx}} + b_{12}\overrightarrow{U_{2xy}} = \overrightarrow{0}, \qquad (5.20) \\
& \overrightarrow{V_2} + (c_4 - \chi'' c_{10})\overrightarrow{U_{1x}} + c_8\overrightarrow{U_{1xx}} + c_{12}\overrightarrow{U_{1xy}} \\
& \qquad +(d_4 - \chi'' d_{10})\overrightarrow{U_{2x}} + d_8\overrightarrow{U_{2xx}} + d_{12}\overrightarrow{U_{2xy}} = \overrightarrow{0}, \qquad (5.21) \\
& \Delta_1 = 0; \qquad \Delta_2 = 0. \qquad (5.22)
\end{aligned}
$$

The system of equations (5.20) and (5.21) can be solved separately. Notice that in (5.20) or (5.21), there are 18 unknowns and 12 equations. The optimization method (cf. [6, 17]) is used to solve the system of equations. Once the coefficients α_k, β_k, γ_k, and δ_k are obtained, the correction terms $C^1_{i,j}$ and $C^2_{i,j}$ can be obtained directly from (5.22) as follows:

$$
\begin{aligned}
C^1_{i,j} &= (a_4 - a_{10}\cdot\chi''(0))w_{1\xi_0} + a_8 w_{1\xi\xi_0} + a_{12} w_{1\xi\eta_0} \\
&\quad +(b_4 - b_{10}\cdot\chi''(0))w_{2\xi_0} + b_8 w_{2\xi\xi_0} + b_{12} w_{2\xi\eta_0}, \\
C^2_{i,j} &= (c_4 - c_{10}\cdot\chi''(0))w_{1\xi_0} + c_8 w_{1\xi\xi_0} + c_{12} w_{1\xi\eta_0} \\
&\quad +(d_4 - d_{10}\cdot\chi''(0))w_{2\xi_0} + d_8 w_{2\xi\xi_0} + d_{12} w_{2\xi\eta_0}.
\end{aligned} \tag{5.23}
$$

To make the system of finite difference equations better conditioned, we impose the sign restriction on the coefficients α_k in (5.20) and the coefficients δ_k in (5.21)

$$
\alpha_5 < 0 \quad \text{if } k = 5, \qquad \alpha_k \geq 0 \quad \text{otherwise}, \tag{5.24}
$$

$$\delta_5 < 0 \quad \text{if } k = 5, \qquad \delta_k \ge 0 \quad \text{otherwise.} \tag{5.25}$$

We also impose the following restrictions for all the coefficients:

$$-1.1 * bnd \le \alpha_k, \beta_k, \gamma_k, \delta_k \le 1.1 * bnd \qquad (k = 1, 2, \cdots, 9), \tag{5.26}$$

where

$$bnd = \max\left(\frac{2(3-4\nu^-)}{h^2}, \frac{2(3-4\nu^+)}{h^2}, \frac{1}{4h^2}\right). \tag{5.27}$$

With these restrictions, the solution for α_k, β_k, γ_k, and δ_k obtained by the optimization method are of order $\mathcal{O}(1/h^2)$.

Once we have all the coefficients in (2.1), we can set up a large system of linear equations (2.2), (5.1) and (2.6), (2.7) with $2(m-1)(n-1)$ unknowns. To solve such a system of linear equations, we have tested SOR, preconditioned GMRES(m), and Bi-CGSTAB methods. Comparisons among these methods show that the preconditioned GMRES(m) [4, 8] and preconditioned Bi-CGSTAB [4, 7, 14] are almost equally successful. We will give some results in the next section using the GMRES(m) method with a diagonal preconditioning.

6 Numerical Examples

We have performed a number of numerical experiments on Sun's Ultra-10 workstations. In these numerical experiments, the computational domain is a rectangular region with either an ellipse or a circle interface within the domain.

We present two numerical examples. We define the following quantities

$$||E_n^{(1)}||_\infty = \max_{i,j} ||u_1(i,j) - U_{1ij}||, \qquad \text{Ratio}_1 = \frac{||E_n^{(1)}||_\infty}{||E_{2n}^{(1)}||_\infty},$$

$$||E_n^{(2)}||_\infty = \max_{i,j} ||u_2(i,j) - U_{2ij}||, \qquad \text{Ratio}_2 = \frac{||E_n^{(2)}||_\infty}{||E_{2n}^{(2)}||_\infty}.$$

They are used in Table 6.1 and Table 6.2.

Example 1. In this example, the domain is $\Omega = (-1/2, 12) \times (-1/2, 1/2)$ and the interface is defined by $x^2 + 4y^2 = 0.35^2$. The values of the Poisson ratio and the shear modulus are, respectively,

$$\nu = \begin{cases} \nu^+ = 0.20 & \text{in } \Omega^+ \\ \nu^- = 0.24 & \text{in } \Omega^- \end{cases}, \qquad \mu = \begin{cases} \mu^+ = 1{,}500{,}000 & \text{in } \Omega^+ \\ \mu^- = 2{,}000{,}000 & \text{in } \Omega^- \end{cases}.$$

The Dirichlet boundary condition and the interface conditions are determined from the following exact solution:

$$u_1(x,y) = \begin{cases} xy + \sin(1 + x^2 + y^2) - 3\,x^2 + y^2 & \text{in } \Omega^+, \\ xy + \sin(1 + x^2 + y^2) - 2\,x^2 + 5\,y^2 - 4 & \text{in } \Omega^-, \end{cases}$$

$$u_2(x,y) = \begin{cases} \cos(1 + x^2 - y^2) + 5\,x^2y + x^2 - y^2 + 2 & \text{in } \Omega^+, \\ \cos(1 + x^2 - y^2) + 5\,x^2y + 3\,x^2 + 7\,y^2 - 6 & \text{in } \Omega^-. \end{cases}$$

Table 6.1 gives the grid refinement analysis, where m is the number of iterations before restarting in the GMRES method. Second order accuracy is achieved since the error ratios approach number four. Figure 6.1 shows plots of the computed solutions u_1 and u_2 and the maximum error with the number of subintervals along each side $n = 40$ and $m = 30$.

n	$\|\|E_n^{(1)}\|\|_\infty$	Ratio_1	$\|\|E_n^{(2)}\|\|_\infty$	Ratio_2	m	iterations
20	2.5749e-3	*	1.9612e-3	*	20	174
40	5.0891e-4	5.0595	4.1027e-4	4.7802	20	429
80	1.4667e-4	3.4699	9.4880e-5	4.3241	25	1681
160	3.6265e-5	4.0443	2.3726e-5	3.9990	30	2873
320	8.6890e-6	4.1737	5.4118e-6	4.3841	32	8831

Table 6.1: The grid refinement analysis for Example 1.

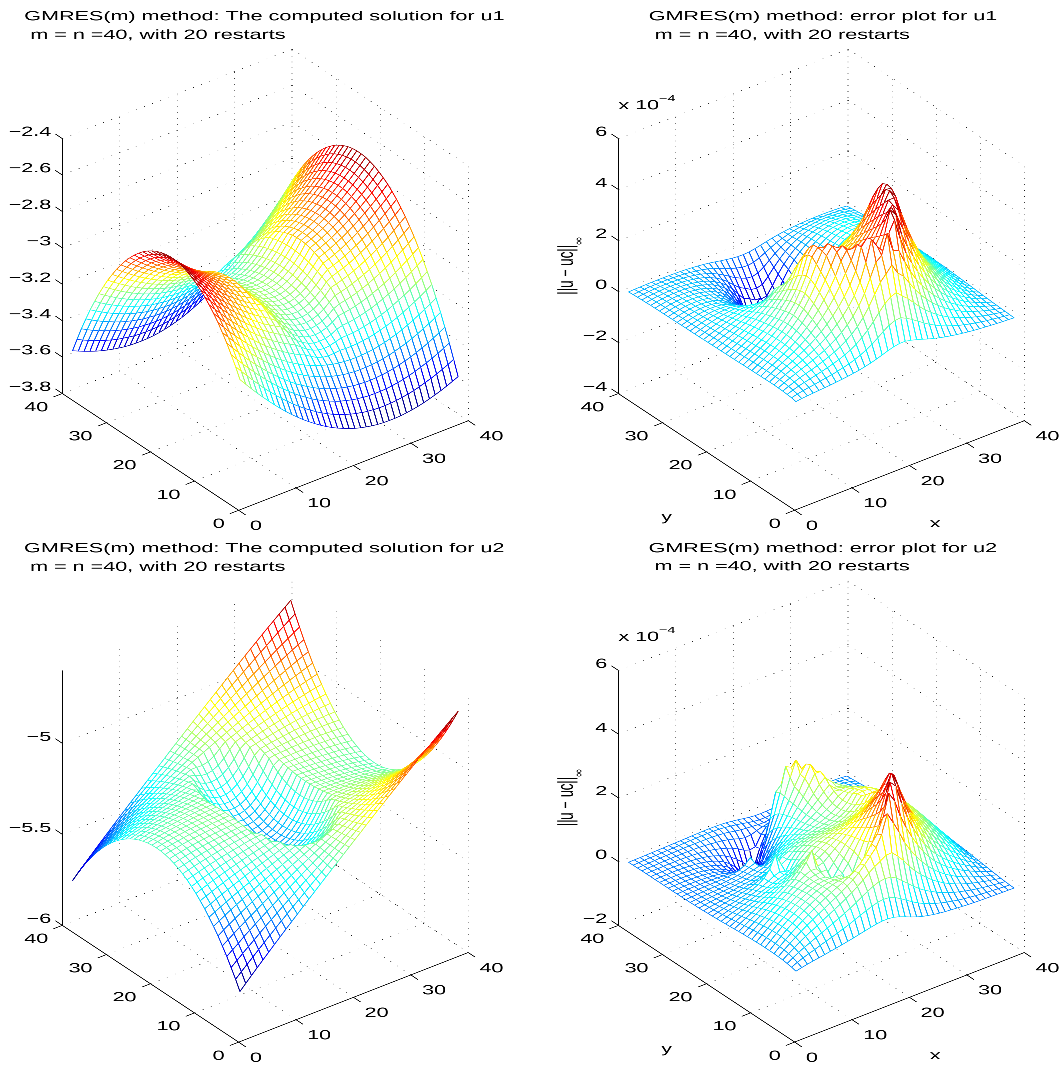

Figure 6.1: The computed solution and the maximum error of Example 1.

Example 2. In this example, the computational domain is $\Omega = (-1, 1) \times (-1, 1)$ and the interface is the circle $x^2 + y^2 = 1/4$. The Dirichlet boundary condition and the interface condition are obtained by the following exact solution:

$$u_1(x, y) = \begin{cases} -(r^4 + c_0 \log(2r))/10 - r_0^2 + (r_0^4 + c_0 \log(2r_0))/10 & \text{in } \Omega^+, \\ -r^2 & \text{in } \Omega^-, \end{cases}$$

$$u_2(x, y) = \begin{cases} \ln(1 + x^2 + 3y^2) + \sin(xy) - 4r^2 + 4r_0^2 & \text{in } \Omega^+, \\ \ln(1 + x^2 + 3y^2) + \sin(xy) & \text{in } \Omega^-, \end{cases}$$

where $r_0 = 0.5$, $c_0 = -0.1$, and $r = \sqrt{x^2 + y^2}$. The values of the Poisson ratio and the shear modulus are, respectively,

$$\nu = \begin{cases} \nu^+ = 0.20 & \text{in } \Omega^+ \\ \nu^- = 0.24 & \text{in } \Omega^- \end{cases}, \qquad \mu = \begin{cases} \mu^+ = 25,000,000 & \text{in } \Omega^+ \\ \mu^- = 30,000,000 & \text{in } \Omega^- \end{cases}.$$

Table 6.2 gives the grid refinement analysis. Second order accuracy again is verified. Figure 6.2 shows plots of the computed solutions u_1 and u_2 and the maximum error with the number of subintervals along each side $n = 40$ and $m = 30$.

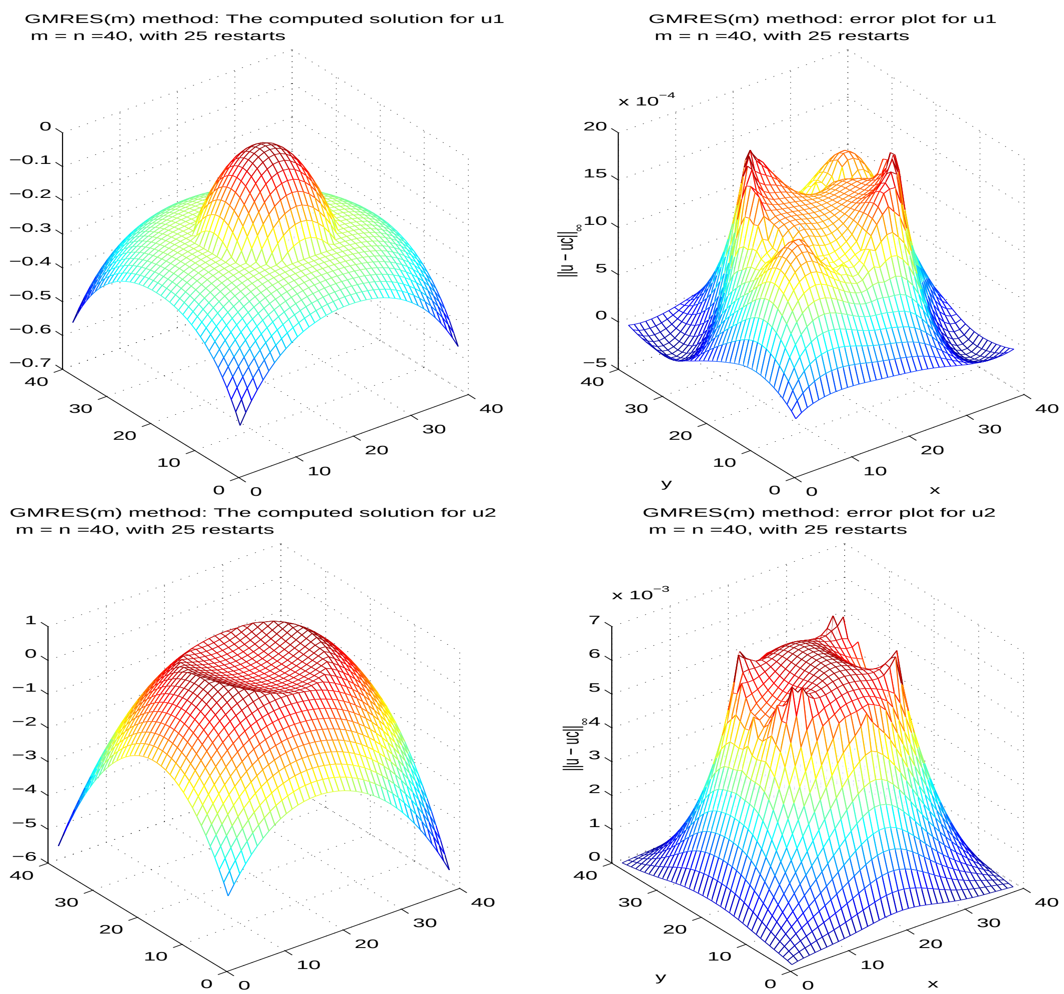

Figure 6.2: Computed solutions and the maximum error for Example 2.

n	$\|\|E_n^{(1)}\|\|_\infty$	Ratio_1	$\|\|E_n^{(2)}\|\|_\infty$	Ratio_2	m	iterations
20	9.8376e-3	*	3.4529e-2	*	20	158
40	1.9038e-3	5.1675	6.6401e-3	5.2001	25	415
80	4.3297e-4	4.3970	1.5098e-3	4.3979	25	1161
160	1.0680e-4	4.0539	3.6397e-4	4.1483	30	3723
320	2.5572e-5	4.1765	8.5800e-5	4.2420	35	8702

Table 6.2: Grid refinement analysis for Example 2.

7 Conclusions

In this paper, we have developed an immersed interface method for elasticity problem with interfaces. We use an optimization method to determine the coefficients of the finite difference equations. We employ the GMRES(m) or Bi-CGSTAB to solve the large, sparse, and non-symmetric linear system equations arising from the discretization of the elasticity equation together with the interface condition. Our numerical experiments confirm that our scheme is second-order accurate.

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References

[1] Yu. A. Amenzade. *Theory of Elasticity.* Mir Publishers Moscow, 1979. Revised from the 1976 Russian edition by M. Konyaeva.

[2] R. J. Asaro and W. A. Tiller. Surface morphology development during stress corrosion cracking: Part I: via surface diffusion. *Metall. Trans.*, 3:1789–1796, 1972.

[3] I. Babuška and T. Strouboulis. *The Finite Element Method and Its Reliability.* Oxford University Press, 2001.

[4] R. Barrett, M. Berry, T. F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. Van der Vorst. *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, 2nd Edition.* SIAM, Philadelphia, PA, 1994.

[5] P.-C. Chou and N. J. Pagano. *Elasticity: Tensor, Dyadic, and Engineering Approaches.* University Series in Basic Engineering. D. Van Nostrand Company, INC., 1967.

[6] J. J. Dongarra, J. R. Bunch, C. B. Moler, and G. W. Stewart. *LINPACK Users' Guide.* SIAM, Philadelphia, 1979.

[7] Diederik R. Fokkema. Enhanced implementation of bicgstab(ℓ) for solving linear systems of equations. Preprint 976, Department of Mathematics, Utrecht University, 1996.

[8] V. Frayssé, L. Giraud, and S. Gratton. A set of GMRES routines for real and complex arithmetics. Technical Report TR/PA/97/49, CERFACS, 1997.

[9] P. L. Gould. *Introduction to Linear Elasticity.* Springer-Verlag, New York, 1994.

[10] M. A. Grinfeld. The stress driven rearrangement instability in ellastic crystals: mathematical models and physical manifestations. *Nonlinear Sci.*, 3:35–83, 1993.

[11] M. A. Haider. *Analytic approximations for the indentation of a thin linear elastic layer and a viscoelastic formulation in finite strain with applications to the mechanics of biological soft tissues.* PhD thesis, Dept. of Mathematical Sciences, Rensselaer Polytechnic Institute, 1996.

[12] T.Y. Hou. Numerical solutions to free boundary problems. *Acta Numerica*, pages 335–415, 1995.

[13] H.-J. Jou, H. Leo, and J. S. Lowengrub. Microstructrual evolution in inhomogeneous elastic media. *J. Comp. Phy.*, 131(1):109–148, 1997.

[14] C. T. Kelley. *Iterative Methods for Linear and Nonlinear Equations.* SIAM, Philadelphia, 1995.

[15] R. J. LeVeque and Z. Li. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources. *SIAM J. Numer. Anal.*, 31:1019–1044, 1994.

[16] Z. Li. *The Immersed Interface Method — A Numerical Approach for Partial Differential Equations with Interfaces.* PhD thesis, University of Washington, 1994.

[17] Z. Li and K. Ito. Maximum principle preserving schemes for interface problems with discontinuous coefficients. *SIAM J. Sci. Comput.*, 23(1):339–361, 2001.

[18] T. Mura. *Micromechanics of defects in solids.* Martinus Nijhoff Publishers, 1982.

[19] C. Pozrikidis. *Boundary Element Methods.* CRC Press, 2002.

[20] I. S. Sokolnikoff. *Mathematical Theory of Elasticity.* McGraw-Hill, 1983.

[21] A. P. Sutton and R. W. Balluffi. *Interfaces in Crystalline Materials.* Oxford University Press, 1996.

[22] A. Wiegmann and K. P. Pube. The explicit-jump immersed interface method: Finite difference methods for PDEs with piecewise smooth solutions. *SIAM J. Numer. Anal.*, 37(3):827–862, 2000.

[23] O. C. Zienkiewicz and R. L. Taylor. *The Finite Element Method: Vol. 2, Solid Mechanics.* Butterworth-Heinemann, 2001.