AMSC 612: Numerical Methods in Partial Differential Equations

Fall Semester, 2002

Instructor: Bo Li

Homework Assignment 1

Due Friday, October 4, 2002

1. Let $\Delta x > 0$, $\Delta t > 0$, and $\nu = \Delta t/(\Delta x)^2$. Show that there exists a constant $C = C(\nu) > 0$ such that
$$\left|1 - 4\nu \sin^2 \frac{k\Delta x}{2} - e^{-k^2\Delta t}\right| \le C(\nu)k^4(\Delta t)^2 \qquad \forall k > 0, \forall \Delta x > 0.$$
Verify that when $\nu = 1/4$ this inequality holds true for $C = 1/2$.
2. Consider the model problem:
$$\begin{aligned} &u_t = u_{xx}, && 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0, && t > 0, \\ &u(x,0) = u_0(x), && 0 \le x \le 1, \end{aligned} \tag{1}$$
where u_0 is given. For $\theta \in [0,1]$, the θ-scheme on a uniform grid with space and time steps Δx and Δt, respectively, is defined by
$$u_j^{n+1} - u_j^n = \nu\left[\theta\delta_x^2 u_j^{n+1} + (1-\theta)\delta_x^2 u_j^n\right] \qquad j = 1, \cdots, J-1;\ n = 0, \cdots, N,$$
where $\nu = \Delta t/(\Delta x)^2$, $\delta_x^2 v(x,t) = \delta_x(\delta_x v(x,t))$ for any function $v = v(x,t)$, and
$$\delta_x v(x) = v\left(x + \frac{\Delta x}{2}, t\right) - v\left(x - \frac{\Delta x}{2}, t\right).$$
Define the truncation error for this of scheme, calculate the leading terms of the truncation error, and prove the consistency of the scheme.
3. Consider the finite difference discretization of the model problem (1) on a uniform grid with space and time steps Δx and Δt, respectively. Denote $\nu = \Delta t/(\Delta x)^2$.
   (1) Show the that scheme
   $$u_j^{n+1} - u_j^{n-1} = \frac{1}{3}\nu\left(\delta_x^2 u_j^{n+1} + \delta_x^2 u_j^n + \delta_x^2 u_j^{n-1}\right)$$
   is stable for all values of ν.
   (2) Show the that scheme
   $$u_j^{n+1} - u_j^{n-1} = \frac{1}{6}\nu\left(\delta_x^2 u_j^{n+1} + 4\delta_x^2 u_j^n + \delta_x^2 u_j^{n-1}\right)$$
   is unstable for all values of ν.

4. On uniform grids with space and and time steps Δx and Δt, respectively, use the explicit scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\left(u_{j+1}^n - u_j^n\right) p_{j+1/2} - \left(u_j^n - u_{j-1}^n\right)\right) p_{j-1/2}}{(\Delta x)^2}$$

to solve the initial-boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right), \qquad 0 < x < 1,\ t > 0,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0,$$
$$u(x,0) = u_0(x), \qquad 0 \le x \le 1,$$

Find the leading terms in the truncation error of the scheme, and deduce a bound on the global error in terms of bounds on the derivatives of the solution u and the function p, under the condition $0 < p(x)\Delta t \le (\Delta x)^2/2$.

5. A two-d problem: ADI method—analysis and implementation.

6. Consider the initial-boundary-value problem

$$h_t = \left((h_x)^3 - h_x - h_{xxx}\right)_x \qquad \forall (x,t) \in (0,12) \times (0,70],$$
$$h(x,t) \text{ is periodic in } x \text{ with period } = 12 \qquad \forall t \in [0,70],$$
$$h(x,0) = 0.1\left(\sin\frac{\pi x}{2} + \sin\frac{2\pi x}{3} + \sin \pi x\right) \qquad \forall x \in [0,12].$$

(1) Design an implicit finite difference scheme to discretize the model, treating the nonlinear term explicitly.

(2) Write a Matlab code to implement the scheme.

(3) Use the Matlab code to compute numerical solutions. Plot the numerical approximation of the solution and that of the standard deviation of the solution defined by

$$w(t) = \sqrt{\frac{1}{L}\int_0^L \left[h(x,t) - \bar{h}(t)\right]^2 dx}, \qquad \text{where } \bar{h}(t) = \frac{1}{L}\int_0^L h(x,t)\,dx$$

where $L = 12$.