AMSC 612: Numerical Methods in Partial Differential Equations

Fall Semester, 2002

Instructor: Bo Li

Homework Assignment 2

Due Wednesday, November 6, 2002

1. For each of the following two problems

$$
\begin{aligned}
&u_t + a(x)u_x = 0, \qquad 0 < x < 1,\ t > 0, \\
&u(x,0) = x(1-x), \qquad 0 \le x \le 1,
\end{aligned}
$$

where $a(x) = x - 1/2$, and

$$
\begin{aligned}
&u_t + a(x)u_x = 0, \qquad 0 < x < 1,\ t > 0, \\
&u(x,0) = x(1-x), \qquad 0 \le x \le 1, \\
&u(0,t) = u(1,t) = 0, \qquad t > 0,
\end{aligned}
$$

where $a(x) = 1/2 - x$:
(1) Sketch the characteristics for the equation in the problem;
(2) Set up the upwind scheme with a uniform spatial grid $\{x_j = j\Delta x : j = 0, 1, \cdots, J\}$ for an integer $J \ge 1$;
(3) Derive the local truncation error;
(4) Derive the global error bounds for both even and odd J;
(5) Sketch the development of the solution.

2. Determine the coefficients c_{-1}, c_0, and c_1 so that the finite difference scheme

$$u_j^{n+1} = c_{-1}u_{j-1}^n + c_0 u_j^n + c_1 u_{j+1}^n$$

for the equation

$$u_t + a u_x = 0,$$

where $a > 0$ is a constant, agrees with the Taylor expansion of $u(x_j, t_{n+1})$ to as high order as possible. Verify that the result is the Lax-Wendroff scheme.

3. For each of the upwind, Lax-Wendroff, and leap-frog schemes applied to the initial-value problem of the equation

$$u_t + a u_x = 0,$$

where $a > 0$ is a constant, find out the amplification factor $\lambda = \lambda(k)$ with k a wavenumber and the leading terms (up to ξ^3 with $\xi = k\Delta x$) in phase expansions.

4. Consider the initial-boundary-value problem of hyperbolic equation

$$
\begin{aligned}
&u_t + a(x,t)u_x = 0, \qquad x > 0,\ t > 0, \\
&u(x,0) = u_0(x), \qquad x \ge 0, \\
&u(0,t) = 0, \qquad x \ge 0,
\end{aligned}
$$

where

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4},$$
$$u_0(x) = \begin{cases} 1 & \text{if } 0.2 \le x \le 0.4, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Verify that the exact solution of this problem is

$$u(x,t) = u_0\left(x - \frac{t}{1+x^2}\right).$$

(2) Use the upwind scheme to solve the problem with $\Delta t = \Delta x = 0.02$ and $\Delta t = \Delta x = 0.01$, respectively. In each case, show for each value of $t = 0$, 0.1, 0.5, and 1, the graph of both the exact and numerical solutions on the same plot.

(3) Use the Lax-Wendroff scheme to solve the problem with $\Delta t = \Delta x = 0.02$ and $\Delta t = \Delta x = 0.01$, respectively. In each case, show for each value of $t = 0$, 0.1, 0.5, and 1, the graph of both the exact and numerical solutions on the same plot.

5. Prove that the numerical scheme

$$u_j^{n+1} = u_j^n - C_{j-1}^n\left(u_j^n - u_{j-1}^n\right) + D_j^n\left(u_{j+1}^n - u_j^n\right)$$

with

$$C_{j-1}^n \ge 0, \qquad D_j^n \ge 0, \qquad C_j^n + D_j^n \le 1, \qquad \forall j, n$$

is total variation diminishing (TVD), i.e., it satisfies

$$\sum_{j=-\infty}^{\infty} \left|u_j^{n+1} - u_{j-1}^{n+1}\right| \le \sum_{j=-\infty}^{\infty} \left|u_j^n - u_{j-1}^n\right|.$$

Put the upwind and Lax-Wendroff schemes into the above form, and show that the former is TVD when it is stable, while the latter is not TVD for any Δt.