AMSC 612: Numerical Methods in Partial Differential Equations
Fall Semester, 2002
Instructor: Bo Li

Homework Assignment 3

Due Friday, December 6, 2002

1. Let $J \geq 1$ be an integer and $-\pi = x_{-J} < \cdots < x_0 < \cdots < x_J = \pi$ be a uniform grid of $[-\pi, \pi]$ with grid size $\Delta x = \pi/J$ and grid points $x_j = j\Delta x$ ($j = 0, \pm 1, \cdots, \pm J$). Let G_J denote the complex inner product space that consists of all the complex-valued, periodic, grid functions with the inner product

$$\langle U, V \rangle = \Delta x \sum_{j=1-J}^{J} U(x_j)\overline{V}(x_j) \qquad \forall U, V \in G_J.$$

For each wavenumber $k = 1-J, 2-J, \cdots, J$, define the Fourier mode $\Phi_k \in G_J$ by

$$\Phi_k(x_j) = \frac{1}{\sqrt{2\pi}} e^{ikx_j}, \qquad j = 0, \pm 1, \cdots, \pm J.$$

For any $U \in G_J$, define its discrete Fourier transform $\hat{U}$ to be the complex-valued function of wavenumber

$$\hat{U}(k) = \frac{\Delta x}{\sqrt{2\pi}} \sum_{j=1-J}^{J} U(x_j) e^{-ikx_j}, \qquad k = 0, \pm 1, \cdots, \pm J.$$

(1) Show that the system of all the Fourier modes $\{\Phi_k\}_{k=1-J}^{J}$ forms an orthonormal basis of the inner product space G_J.

(2) Show that

$$U(x_j) = \sum_{k=1-J}^{J} \hat{U}(k)\Phi_k(x_j), \qquad j = 0, \pm 1, \cdots, \pm J.$$

(3) Prove the Parseval identity: $\|\hat{U}\| = \|U\|$ for all $U \in G_J$, where $\|\hat{U}\|^2 = \sum_{k=1-J}^{J} |\hat{U}(k)|^2$ and $\|U\|^2 = \langle U, U \rangle$.

2. Let $n \geq 2$ be an integer. For any $v = (v_0, \cdots, v_n) \in \mathbb{R}^{n+1}$ and $w = (w_0, \cdots, w_n) \in \mathbb{R}^{n+1}$, denote

$$\langle v, w \rangle = \sum_{j=1}^{n-1} v_j w_j.$$

Denote also

$$\Delta_+ v_j = v_{j+1} - v_j, \qquad \Delta_- v_j = v_j - v_{j-1}, \qquad \Delta_0 v_j = \frac{1}{2}(v_{j+1} - v_{j-1}).$$

Prove the following formulas of summation by parts:

$$\langle v, \Delta_+ w \rangle + \langle \Delta_- v, w \rangle = v_{n-1}w_n - v_0 w_1;$$

$$\langle v, \Delta_0 w \rangle + \langle \Delta_0 v, w \rangle = \frac{1}{2}\left[(v_{n-1}w_n + v_n w_{n-1}) - (v_0 w_1 + v_1 w_0)\right].$$

3. Consider the finite difference discretiztion with a uniform grid of the one-dimensional diffusion-advection equation

$$u_t + au_x = \epsilon u_{xx}, \tag{1}$$

where $a \neq 0$ and $\epsilon > 0$ are constants. Denote by Δx and Δt the spatial grid size and time step. Denote also the ratios $\nu = |a|\Delta t/\Delta x$ and $\mu = \epsilon \Delta t/(\Delta x)^2$.

   (1) Consider the following scheme that is central in space and forward in time for Eq. (1)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{\Delta_{0i} u_i^n}{\Delta x} = \epsilon \frac{\delta_i^2 u_i^n}{(\Delta x)^2}.$$

      (a) Calculate the amplification factor $\lambda = \lambda(k)$.

      (b) Show that if $\mu \leq 1/2$ then

$$|\lambda(k)|^2 \leq 1 + \frac{a^2}{2\epsilon}\Delta t.$$

      Thus, the von Neumann's stability condition is satisfied. Let $\nu = 1$ and $\mu = 1/4$. What is then $|\lambda(k)|$ for the wavenumber k that satisfies $\sin^2(k\Delta x)/2 = 1/2$? Interpret it in terms of the growth of numerical oscillation.

      (c) Show that $|\lambda(k)| \leq 1$ for all k if and only if $\nu^2 \leq 2\mu \leq 1$.

   (2) Consider a modification of the above scheme that uses the up-wind scheme to discretize the advection term au_x but uses the same discretization for the terms u_t and ϵu_{xx}.

      (a) Calculate the amplification factor $\lambda = \lambda(k)$.

      (b) Show that $|\lambda(k)| \leq 1$ for all k if and only if $\nu^2 \leq \nu + 2\mu \leq 1$.

4. Suppose that the Lax-Wendroff method is applied on a uniform grid to the advection equation $u_t + au_x = 0$ with constant $a > 0$. Let $\nu = a\Delta t/\Delta x$, where Δx and Δt are the spatial grid size and time step. Show that, over the whole real line,

$$\|u^{n+1}\|^2 = \|u^n\|^2 - \frac{1}{2}\nu^2(1-\nu^2)\left(\|\Delta_- u^n\|^2 - \langle \Delta_- u^n, \Delta_+ u^n \rangle\right),$$

and hence deduce the stability condition.

5. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and $f \in C(\bar{\Omega})$. Use the standard five-point scheme to discretize the boundary-value problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

with a uniform grid in which the grid size is $\Delta x = \Delta y = 1/n$ for some integer $n \geq 2$.

   (1) Write a simple Matlab code for solving the linear system using both Gauss-Jacobi and Gauss-Seidel iterative methods. Test the code with the exact solution $u(x, y) = \sin \pi x \, \sin \pi y$ for the related f.

   (2) Order all the interior grid points in a natural way, e.g., from bottom to top and from left to right, and write down the coefficient matrix and the right-hand side vector for the resulting linear system of equations.