AMSC 614: Mathematics of the Finite Element Method
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Instructor: Bo Li
Department of Mathematics
University of Maryland, College Park

Exercise Problems

Chapter 0. Introduction

1. Let H be a real Hilbert space and denote its inner product by $\langle \cdot, \cdot \rangle$. Let $n \geq 1$ be an integer and $\phi_1, \cdots, \phi_n$ be n elements in H. Show that the *Gram matrix* $G(\phi_1, \cdots, \phi_n)$ for $\phi_1, \cdots, \phi_n$, defined by $G(\phi_1, \cdots, \phi_n) = (\langle \phi_i, \phi_j \rangle)$, is symmetric positive semi-definite, and that it is positive definite if and only if $\phi_1, \cdots, \phi_n$ are linearly independent.
2. Let $f \in C[0,1]$. Assume that
$$\int_0^1 f(x)\phi(x)\,dx = 0 \qquad \forall \phi \in C_0^\infty(0,1).$$
Prove that $f(x) = 0$ for all $x \in [0,1]$.
3. Let $f \in C[0,1]$ and consider the two-point boundary value problem
$$\begin{cases} -u'' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$
Let $n \geq 1$ be an integer, $h = 1/n$, and $x_i = ih$ $(i = 0, \cdots, n)$. Let
$$V_h = \{v_h \in C[0,1] : v_h(0) = v_h(1) = 0,\, v_h|_{[x_{i-1},x_i]} \in P_1([x_{i-1}, x_i]),\, i = 1, \cdots, n\}$$
be the continuous piecewise linear finite element space. Let $u_h \in V_h$ be the corresponding finite element approximation to the exact solution u, defined by
$$\int_0^1 [u'(x) - u_h'(x)]\; v_h'(x)\,dx = 0 \qquad \forall v_h \in V_h.$$
Show that
$$u_h(x_i) = u(x_i), \qquad i = 0, \cdots, n.$$

Chapter 1. Sobolev Spaces and Elliptic Boundary Value Problems

1. Let H be a real Hilbert space. Denote by $\langle \cdot, \cdot \rangle$ its inner product and by $\| \cdot \|$ the corresponding norm. Let K be a non-empty, closed, convex subset of H. Let $x \in H$. Prove that there exists a unique $u \in K$ such that

$$\|u - x\| = \inf_{v \in K} \|v - x\|.$$

The element $u \in K$ is called the *projection of x onto K* and is denoted by $u = P_K x$. Prove also the following.

   (1) In general, the projection $u = P_K x$ is characterized by $u \in K$ and

   $$\langle x - u, \, v - u \rangle \leq 0 \qquad \forall v \in K.$$

   (2) If K is a non-empty, closed, convex cone of H with vertex 0, then the projection $u = P_K x$ is characterized by $u \in K$ and

   $$\begin{aligned} &\langle x - u, \, v \rangle \leq 0 \qquad \forall v \in K, \\ &\langle x - u, \, u \rangle = 0. \end{aligned}$$

   (3) If K is a closed subspace of H, then the projection $u = P_K x$ is characterized by $u \in K$ and

   $$\langle x - u, \, v \rangle = 0 \qquad \forall v \in K.$$

2. Let $\beta \in \mathbb{R}$ be such that $1/2 \leq \beta < \infty$. Define using the polar coordinate

$$\Omega_\beta = \left\{ (r, \theta) : 0 < r < 1, \, 0 < \theta < \frac{\pi}{\beta} \right\}$$

and

$$u_\beta(r, \theta) = r^\beta \sin \beta\theta \qquad \forall (r, \theta) \in \Omega_\beta.$$

   (1) Find all values of the integer $k \geq 0$ and the extended real number $p \in [1, \infty]$ such that $u_\beta \in W^{k,p}(\Omega_\beta)$.

   (2) Let $\beta = 1/2$. The domain $\Omega_{1/2}$ is called a *slit domain*. Find all integers $k \geq 1$ and extended real numbers $p \in [1, \infty]$ such that $u_{1/2} \in W^{k,p}(\Omega_{1/2})$.

   (3) Let $\beta = 2/3$. The domain $\Omega_{2/3}$ is called a *reentrant corner*. Find all integers $k \geq 1$ and extended real numbers $p \in [1, \infty]$ such that $u_{2/3} \in W^{k,p}(\Omega_{2/3})$.

3. Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$ with $n \geq 2$. Define $u : \Omega \to \mathbb{R}$ by

$$u(x) = \log \log \left(1 + \frac{1}{|x|} \right) \qquad \forall x \in \Omega.$$

Show that $u \in W^{1,n}(\Omega)$ but $u \notin L^\infty(\Omega)$.

4. Let p and r be two real numbers such that $2 < p < r$. Let

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \, |y| < x^r\}.$$

Define $u : \Omega \to \mathbb{R}$ by $u(x, y) = x^{-1/p}$ for $(x, y) \in \Omega$. Show that $u \in W^{1,p}(\Omega)$ but $u \notin L^\infty(\Omega)$. Dose this contradict the Sobolev embedding for the case $p > n$ with $n = 2$?

5. Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a bounded domain with a Lipschitz boundary.

(1) Show that

$$\|\Delta u\|_{L^2(\Omega)} = \|D^2 u\|_{L^2(\Omega)} \qquad \forall u \in H_0^2(\Omega),$$

where Δ is the n-dimensional Laplacian and

$$\|D^2 u\|_{L^2(\Omega)}^2 = \sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u(x)|^2 \, dx.$$

(2) Show that $u \longmapsto \|\Delta u\|_{L^2(\Omega)}$ defines a norm of $H_0^2(\Omega)$ and this norm is equivalent to the usual $H^2(\Omega)$-norm of $H_0^2(\Omega)$.

6. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Set $\mathcal{E}(u) = (\nabla u + (\nabla u)^T)/2$ for any $u \in H^1(\Omega; \mathbb{R}^3)$.

(1) Prove the identity

$$\|\nabla u\|_{L^2(\Omega)}^2 + \|\mathrm{tr}\,(\nabla u)\|_{L^2(\Omega)}^2 = 2\|\mathcal{E}(u)\|_{L^2(\Omega)}^2 \qquad \forall u \in H_0^1(\Omega; \mathbb{R}^3).$$

(2) Prove Korn's first inequality: there exists a constant $C = C(\Omega) > 0$ such that

$$\|u\|_{H^1(\Omega)} \le C\|\mathcal{E}(u)\|_{L^2(\Omega)} \qquad \forall u \in H_0^1(\Omega; \mathbb{R}^3).$$

(3) Let $\Gamma_0 \subset \partial\Omega$ be dS-measurable with a positive dS-measure. Let

$$H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) = \{u \in H^1(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \Gamma_0\}.$$

Prove that there exists a constant $C = C(\Omega, \Gamma_0) > 0$ such that

$$\|u\|_{H^1(\Omega)} \le C\|\mathcal{E}(u)\|_{L^2(\Omega)} \qquad \forall u \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3).$$

(Hint: use Korn's second inequality, and show that $\mathcal{E}(u) = 0$ in Ω implies that $u(x) = a \times ox + b$ for some vectors $a, b \in \mathbb{R}^3$.)

7. Let $n \ge 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$. Let $a_{ij} \in L^\infty(\Omega)$ for all $i, j = 1, \cdots, n$, and assume that there exists a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \qquad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n.$$

Let $b \in L^\infty(\Omega)$ with $b \ge 0$ a.e. Ω and $f \in L^2(\Omega)$. Moreover, let $\Gamma_0 \subset \Gamma$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$ be both dS-measurable subsets of Γ with positive dS-measure. Let $g \in L^2(\Gamma_1)$. Prove the following.

(1) There exists a unique $u \in V$ such that

$$A(u, v) = F(v) \qquad \forall v \in V,$$

where

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\},$$

$$A(u, v) = \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}u(x)\partial_{x_j}v(x) + b(x)u(x)v(x) \right] dx,$$

$$F(v) = \int_\Omega f(x)v(x)\,dx + \int_{\Gamma_1} g(x)v(x)\,dS.$$

(2) If in addition $a_{ij} \in W^{1,\infty}(\Omega)$ $(i,j = 1, \cdots, n)$ and $u \in C^2(\bar{\Omega})$, then u solves the boundary value problem

$$\begin{cases} -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}\partial_{x_i} u) + b\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \sum_{i,j=1}^{n} a_{ij}\partial_{x_i} u\,\nu_j = g & \text{on } \Gamma_1, \end{cases}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outer normal along the boundary $\partial\Omega$.

8. Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a bounded domain with a Lipschitz boundary $\partial\Omega$. Let $b \in L^\infty(\Omega)$ with $b \geq 0$ a.e. Ω and $f \in L^2(\Omega)$. Let $c \in L^\infty(\Gamma)$ with $c \geq 0$ a.e. Γ and $g \in L^2(\Gamma)$. Suppose either $\text{meas}\{x \in \Omega : b(x) > 0\} > 0$ or $dS\text{-meas}\{x \in \partial\Omega : c(x) > 0\} > 0$. Prove the following.

(1) There exists a unique $u \in H^1(\Omega)$ such that

$$A(u,v) = F(v) \qquad \forall v \in H^1(\Omega),$$

where

$$A(u,v) = \int_\Omega \Big[\nabla u(x) \cdot \nabla v(x) + b(x)u(x)v(x)\Big]\,dx + \int_{\partial\Omega} c(x)u(x)v(x)\,dS,$$

$$F(v) = \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} g(x)v(x)\,dS.$$

(2) If in addition $u \in C^2(\bar{\Omega})$, then u solves the following boundary value problem

$$\begin{cases} -\Delta u + b(x)u = f(x) & \text{in } \Omega, \\ \partial_\nu u + c(x)u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outer normal along the boundary $\partial\Omega$.

9. Let $n \geq 2$ be an integer. Let Ω, Ω_1, and Ω_2 be bounded Lipschitz domains in $\mathbb{R}^n$ such that $\Omega_i \subset \Omega$ $(i = 1,2)$, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, and each $\Gamma_i := \partial\Omega_i \cap \partial\Omega$ $(i = 1,2)$ has a positive dS-measure. Denote $S = \partial\Omega_1 \cap \partial\Omega_2$. Consider the interface boundary value problem

$$\begin{cases} -\nabla \cdot a\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ a_2\partial_\nu u = g & \text{on } \Gamma_2, \\ [u] = [a\partial_\nu u] = 0 & \text{on } S, \end{cases}$$

where

$$a(x) = \begin{cases} a_1 & \text{if } x \in \Omega_1, \\ a_2 & \text{if } x \in \Omega_2, \end{cases}$$

and a_1 and a_2 are two distinct, positive, real numbers, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$, ν is the unit exterior normal of $\partial\Omega_2$, and $[\cdot]$ denotes the jump across the interface S.

(1) Find the weak formulation of the boundary value problem.

(2) Prove that the problem in weak formulation has a unique solution.

(3) Prove that the weak solution, if it is smooth enough, solves the boundary value problem.

10. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Show that the boundary value problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution u which is defined by $u \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega v(x)\, dx \qquad \forall v \in H_0^1(\Omega).$$

Show also that the weak solution $u \notin C^2(\bar{\Omega})$.

Chapter 2. Construction of Finite Elements and Finite Element Spaces

1. Let Ω and $\Omega_1, \cdots, \Omega_r$ ($r \geq 2$) be bounded Lipschitz domains in $\mathbb{R}^n$ ($n \geq 1$) such that $\Omega_i \subset \Omega$ ($1 \leq i \leq r$), $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ ($1 \leq i, j \leq r$), and $\bar{\Omega} = \cup_{i=1}^r \bar{\Omega}_i$.
   (1) Suppose that $u \in L^2(\Omega)$ and that $u|_{\Omega_i} \in H^1(\Omega_i)$ ($1 \leq i \leq r$). Show that $u \in H^1(\Omega)$ if and only if $u|_{\Omega_i} = u|_{\Omega_j}$ on $\partial\Omega_i \cap \partial\Omega_j$ ($1 \leq i, j \leq r$).
   (2) Suppose that $u \in L^2(\Omega)$ and that $u|_{\Omega_i} \in H^2(\Omega_i)$ for all $i \in \{1, \cdots, r\}$. Show that $u \in H^2(\Omega)$ if and only if both $u|_{\Omega_i} = u|_{\Omega_j}$ and $\nabla u|_{\Omega_i} = \nabla u|_{\Omega_j}$ on $\partial\Omega_i \cap \partial\Omega_j$ ($1 \leq i, j \leq r$).
2. Let X be a real vector space with a finite dimension $d \geq 1$. Denote by X' the set of all linear functionals from X to $\mathbb{R}$. Prove the following.
   (1) The set X' is also a vector space with the same dimension d.
   (2) Let $\{\Phi_1, \cdots \Phi_d\} \subset X'$. Then, the following are equivalent: (a) $\{\Phi_1, \cdots \Phi_d\}$ is a basis for X'; (b) $\cap_{i=1}^d \text{Ker}\, \Phi_i = \{0\}$; (c) There exist $\phi_i \in X$, $i = 1, \cdots, d$, such that $\Phi_i(\phi_j) = \delta_{ij}$, $i, j = 1, \cdots, d$.
   (3) Any basis for X has a unique dual basis for X', and any basis for X' has a unique dual basis for X.
   (4) If $\{\phi_1, \cdots, \phi_d\} \subset X$ and $\{\Phi_1, \cdots, \Phi_d\} \subset X'$ are a pair of dual bases, then

$$f = \sum_{i=1}^d \Phi_i(f)\phi_i \qquad \forall f \in X,$$

$$F = \sum_{i=1}^d F(\phi_i)\Phi_i \qquad \forall F \in X'.$$

3. Let $T = \Delta z_1 z_2 z_3 \subset \mathbb{R}^2$ be a triangle. Let $\lambda_i = \lambda_i(x) \in P_1(T)$ ($i = 1, 2, 3$) be the barycentric coordinates of $x \in T$ such that $\lambda_i(z_j) = \delta_{ij}$ ($i, j = 1, 2, 3$).
   (1) Let (ijk) be any permutation of (123). Show that $\nabla\lambda_i \cdot \nabla\lambda_j \leq 0$ if and only if the angle $\angle z_i z_k z_j \leq 90°$. Moreover, $\nabla\lambda_i \cdot \nabla\lambda_j = 0$ if and only if the angle $\angle z_i z_k z_j = 90°$.
   (2) Let α, β, and γ be any nonnegative integers. Show that

$$\int_T \lambda_1^\alpha(x)\lambda_2^\beta(x)\lambda_3^\gamma(x)\, dx = 2|T| \frac{\alpha!\, \beta!\, \gamma!}{(\alpha + \beta + \gamma + 2)!},$$

   where $|T|$ denotes the area of T.

(3) Find the shape functions for the cubic triangular finite element (T, P_3, Σ), where $\Sigma = \{p(z) : z \in L_3(T)\}$ and $L_3(T)$ is the set of all principle lattice points of T of order 3.

4. Let $n \geq 2$ be an integer and $T = S(z_0, \cdots, z_n) \subset \mathbb{R}^n$ an n-dimensional simplex. Let $\lambda_0(x), \cdots, \lambda_n(x)$ be the barycentric coordinates of $x \in T$ such that $\lambda_i(z_j) = \delta_{ij}$ $(i, j = 0, \cdots, n)$.

   (1) Prove the identity

$$p = \sum_{i=0}^{n} \frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)p(z_i) + \sum_{0\leq i,j\leq n,\, i\neq j} \frac{9}{2}\lambda_i\lambda_j(3\lambda_i - 1)p(z_{iij}) + \sum_{0\leq i<j<k\leq n} 27p(z_{ijk}) \qquad \forall p \in P_3,$$

   where $z_{ijk} = (z_i + z_j + z_k)/3$ and $z_{iij} = (2z_i + z_j)/3$.

   (2) For each triple of integers (i, j, k) with $0 \leq i < j < k \leq n$, let

$$\psi_{ijk}(p) = 12p(z_{ijk}) + 2\sum_{l=i,j,k} p(z_l) - 3\sum_{l,m=i,j,k,\, l\neq m} p(z_{llm})$$

   and

$$P_3' = \{p \in P_3 : \psi_{ijk}(p) = 0 \text{ for all } i, j, k \text{ with } 0 \leq i < j < k \leq n\}.$$

   Prove that $P_2 \subset P_3'$ and that $(T, P_3'(T), \Sigma_T)$ is a finite element, where

$$\Sigma_T = \{p(z_i)\,(0 \leq i \leq n),\ p(z_{iij})\,(0 \leq i, j \leq n,\ i \neq j)\}.$$

5. Let n and k be two positive integers. Prove that

$$p(x_1, \cdots, x_n) = \sum_{0\leq i_l\leq k,\, 1\leq l\leq n} \left(\prod_{m=1}^{n} \prod_{0\leq i_m'\leq k,\, i_m'\neq i_m} \frac{kx_m - i_m'}{i_m - i_m'} \right) p\left(\frac{i_1}{k}, \cdots, \frac{i_n}{k}\right) \quad \forall p \in Q_k.$$

6. (The Q_2' element.) Let $T = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. Denote by z_1, z_2, z_3, and z_4 the vertices of T, ordered counterclockwise starting from the lower left one. Denote by z_5, z_6, z_7, and z_8 the midpoints of the four edges of T, ordered counterclockwise starting from the one of the bottom edge. Denote by $z_9 = (0, 0)$ the center of T. Define

$$Q_2' = \left\{ p \in Q_2 : 4p(z_9) + \sum_{i=1}^{4} p(z_i) - 2\sum_{i=5}^{8} p(z_i) \right\}.$$

   Finally, define $\Sigma = \{p(z_i) : i = 1, \cdots, 8\}$.

   (1) Show that $P_2 \subset Q_2'$.

   (2) Show that (T, Q_2', Σ) defines a finite element.

   (3) Find the shape functions of the finite element (T, Q_2', Σ).

7. Let $T = \Delta z_1 z_2 z_3$ be a triangle and denote $z_{ijk} = (z_i + z_j + z_k)/3$ and $z_{iij} = (2z_i + z_j)/3$ for $i, j, k = 1, 2, 3$. Let $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ be the barycentric coordinates of $x \in T$ such that $\lambda_i(z_j) = \delta_{ij}$ $(i, j = 1, 2, 3)$.

(1) Define

$$\begin{aligned} P_T &= \text{span}\,\{\lambda_i^2\,(1 \le i \le 3),\ \lambda_i\lambda_j\,(1 \le i < j \le 3),\ \lambda_1\lambda_2\lambda_3\}, \\ \Sigma_T &= \{p(z_i)\,(1 \le i \le 3),\ p(z_{ij})\,(1 \le i < j \le 3),\ p(z_{123})\}. \end{aligned}$$

Prove that (T, P_T, Σ_T) defines a finite element and that $P_2(T) \subset P_T$.

(2) Define

$$\begin{aligned} P_T = \text{span}\,\{&\lambda_i^3\,(1 \le i \le 3),\ \lambda_i^2\lambda_j,\ \lambda_i\lambda_j^2\,(1 \le i < j \le 3), \\ &\lambda_i^2\lambda_{i+1}\lambda_{i+2}\,(1 \le i \le 3\,(\text{mod}\,3))\}, \\ \Sigma_T = \{&p(z_i)\,(1 \le i \le 3),\ p(z_{iij})\,(1 \le i, j \le 3,\ i \ne j),\ p(z_i^*)\,(1 \le i \le 3)\}, \end{aligned}$$

where

$$z_i^* = \frac{3}{2}(1-\alpha)z_{123} + \frac{1}{2}(3\alpha - 1)z_i, \qquad 1 \le i \le 3,$$

and $\alpha \in (0,1)$, $\alpha \ne 1/3$. Prove that (T, P_T, Σ_T) defines a finite element and that $P_3(T) \subset P_T$.

(3) Let P_T be the same as in (2). Define

$$\Sigma_T = \{p(z_{ij})\,(1 \le i < j \le 3),\ p(z_{ij}^*)\,(1 \le i, j \le 3,\ i \ne j),\ p(z_i^{**})(1 \le i \le 3)\},$$

where

$$\begin{aligned} z_{ij}^* &= \gamma_1 z_i + \gamma_2 z_j, \qquad 1 \le i, j \le 3,\ i \ne j, \\ \gamma_1 &= \frac{1}{2}\left(1 - \sqrt{\frac{3}{5}}\right), \quad \gamma_2 = \frac{1}{2}\left(1 + \sqrt{\frac{3}{5}}\right), \end{aligned}$$

and

$$z_i^{**} = \alpha z_i + \frac{1-\alpha}{2}(z_{i+1} + z_{i+2}), \qquad 1 \le i \le 3\,(\text{mod}\,3),$$

and $\alpha \in (0,1)$, $\alpha \ne 1/3$. (Notice that $\gamma_1, 1/2, \gamma_2$ are the Gaussian quadrature points of the interval $[0,1]$.) Prove that (T, P_T, Σ_T) defines a finite element.

8. Let T be a rectangle with sides parallel to the coordinate axes, $P_T = Q_1(T)$, and $\Sigma_T = \{p(z_i) : 1 \le i \le 4\}$, where z_i $(1 \le i \le 4)$ are the midpoints of edges of T. Show that Σ_T does not determine P_T.

9. (The three-dimensional rotated Q_1 finite element.) Let $T = \prod_{i=1}^3 [a_i, b_i]$,

$$P = \text{span}\left\{1, x_1, x_2, x_3, \left(\frac{x_1}{l_1}\right)^2 - \left(\frac{x_2}{l_2}\right)^2, \left(\frac{x_2}{l_2}\right)^2 - \left(\frac{x_3}{l_3}\right)^2\right\},$$

where $l_i = b_i - a_i$ $(i = 1, 2, 3)$, and $\Sigma = \{p(c_i) : i = 1, \cdots, 6\}$, where $c_1, \cdots, c_6$ are the centers of the six faces of the 3-rectangle T.

(1) Prove that (T, P, Σ) defines a finite element.

(2) Find the shape functions of this finite element.

10. Let $\Omega = (0,1) \times (0,1)$. Fix an integer $N \geq 2$ and decompose Ω into small triangles using the lines $x = i/N$, $y = j/N$, and $x - y = \pm k/N$ $(i, j, k = 0, \cdots, N)$. Denote by $\mathcal{T}_h$ the resulting finite element triangulation of Ω, where $h = \sqrt{2}/N$ is the mesh size. Denote by V_{h0} the Crouzeix-Raviart nonconforming finite element space over the mesh $\mathcal{T}_h$ corresponding to the homogeneous Dirichlet boundary condition. Prove the following discrete Poincaré inequality:

$$\|v_h\|^2_{L^2(\Omega)} \leq 2 \sum_{T \in \mathcal{T}_h} \|\nabla v_h\|^2_{L^2(T)} \qquad \forall v_h \in V_{h0}.$$

11. (The Morley finite element and the associated finite element space.)
    (1) Let $T = \Delta z_1 z_2 z_3 \subset \mathbb{R}^2$ be a triangle. Let $P = P_2(T)$ and $\Sigma = \{p(z_i)\,(1 \leq i \leq 3); \partial_\nu p(z_{ij})\,(1 \leq i < j \leq 3)\}$, where ∂_ν is the exterior normal derivative and $z_{ij} = (z_i + z_j)/2$ $(i, j = 1, 2, 3)$. Prove that the triple (T, P, Σ) defines a finite element.
    (2) Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with a Lipschitz boundary and $\mathcal{T}_h$ a finite element triangulation of Ω. Let V_h be the corresponding Morley finite element space. Show that in general $V_h \not\subset C(\bar{\Omega})$ and $V_h \not\subset H^1(\Omega)$.
    (3) Let V_{h0} be the subspace of V_h consisting of all functions $v_h \in V_h$ such that v_h vanishes on $\mathcal{Z}_h \cap \partial\Omega$ and $\partial_\nu v_h$ on $\mathcal{M}_h \cap \partial\Omega$, where $\mathcal{Z}_h$ and $\mathcal{M}_h$ are the sets of vertices and edge midpoints, respectively, of all triangular elements in the mesh $\mathcal{T}_h$. Define $|v_h|_{2,h}$ by $|v_h|^2_{2,h} = \sum_{T \in \mathcal{T}_h} |v_h|^2_{2,T}$ for any $v_h \in V_{h0}$. Show that $|\cdot|_{2,h}$ is a norm of V_{h0}.
12. (The static condensation of the degrees of freedom in the Hermite triangular finite element approximation.) Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $\mathcal{T}_h$ a finite element triangulation of Ω, and V_h the corresponding Hermite triangular finite element space. Let $\phi_1, \cdots, \phi_M$ be all the global shape functions in V_h corresponding to all the barycenters of triangular elements in $\mathcal{T}_h$ and $\phi_{M+1}, \cdots, \phi_N$ all the other global shape functions in V_h. The finite element solution $u_h = \sum_{i=1}^N \xi_i \phi_i \in V_h$ to a boundary value problem of a nonhomogeneous biharmonic equation on the domain Ω is obtained by solving the linear system

    $$\sum_{j=1}^N A(\phi_i, \phi_j)\xi_j = b_i, \qquad i = 1, \cdots, N,$$

    for some $(b_1, \cdots, b_N) \in \mathbb{R}^N$, where $A(\phi_i, \phi_j) = \displaystyle\int_\Omega \Delta\phi_i \Delta\phi_j \, dx$. Prove that this linear system of N equations for the N unknowns $\xi_1, \cdots, \xi_N$ can be reduced to an equivalent linear system of $N - M$ equations for the $N - M$ unknowns $\xi_{M+1}, \cdots, \xi_N$ corresponding to non-barycentric nodes, by eliminating the M unknowns $\xi_1, \cdots, \xi_M$ corresponding to the barycentric nodes.
13. (Comparison of the finite element method and the finite difference method.) Consider the boundary value problem

    $$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = (0,1) \times (0,1)$ and $f \in C(\bar{\Omega})$. Fix an integer $N \geq 2$.

(1) Discretize the boundary value problem (1) by the five-point finite difference scheme with a uniform gird of Ω of size $1/N$ in both directions.

(2) Decompose Ω into small triangles using the lines $x = i/N$, $y = j/N$, and $x - y = \pm k/N$ $(i, j, k = 0, \cdots, N)$. Denote by $\mathcal{T}_h$ the resulting finite element triangulation of Ω, where $h = \sqrt{2}/N$ is the mesh size. Use the piecewise linear finite element method with the mesh $\mathcal{T}_h$ to discretize the boundary value problem (1).

(3) Choose a suitable ordering of nodal points and use a suitable numerical quadrature for the calculation of the load vector in the finite element discretization to obtain the linear system same as that resulting from the five-point finite difference discretization.

14. Write a computer program to solve numerically the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0,1) \times (0,1)$ and $f(x,y) = 2(x + y - x^2 - y^2)$, using the piecewise linear finite element method. (The exact solution is $u(x,y) = x(1-x)y(1-y)$.)

Chapter 3. Finite Element Interpolation and Approximation

1. Let (T, P, Σ) be an n-dimensional Lagrange finite element with $\dim P = d$, the set of degrees of freedom $\Sigma = \{p(z_i) : 1 \leq i \leq d\}$, and the set of shape functions $\Phi = \{\phi_i : 1 \leq i \leq d\}$, where $\phi_i(z_j) = \delta_{ij}$ $(i, j = 1, \cdots, d)$. Let $\Pi : C(T) \to P$ denote the Lagrange interpolation operator. Assume that $P_k(T) \subseteq P$ for some integer $k \geq 1$. Prove the following.

(1) For any n-dimensional multi-index α with $0 \leq |\alpha| \leq k$,

$$\sum_{i=1}^{d} (z_i - x)^\alpha \phi_i(x) = \delta_{0,|\alpha|} = \begin{cases} 1 & \text{if } |\alpha| = 0 \\ 0 & \text{if } 1 \leq |\alpha| \leq k \end{cases} \qquad \forall x \in T.$$

(2) For any $u \in C^{k+1}(T)$ that

$$(\Pi u - u)(x) = \sum_{i=1}^{d} R_i(u, x)\phi_i(x) \qquad \forall x \in T,$$

where $R_i(u, x)$ for $1 \leq i \leq d$ is the remainder in the Taylor expansion

$$u(z_i) = u(x) + Du(x)(z_i - x) + \cdots + \frac{1}{k!} D^k u(x)(z_i - x)^k + R_i(u, x), \qquad x \in T.$$

(3) For any $u \in C^{k+1}(T)$ and n-dimensional multi-index α with $1 \leq |\alpha| \leq k$ that

$$\partial^\alpha(\Pi u - u)(x) = \sum_{i=1}^{d} R_i(u, x)\partial^\alpha \phi_i(x) \qquad \forall x \in T.$$

2. Let $n \geq 1$ be an integer. Let (T, P, Σ) and $(\hat{T}, \hat{P}, \hat{\Sigma})$ be two n-dimensional, affine-equivalent, Lagrange finite elements. Let $d = \dim P$ and $\Sigma = \{p(z_i) : i = 1, \cdots, d\}$. Let $1 \leq q \leq \infty$. Show that there two positive constants C_1 and C_2, depending only on q and $(\hat{T}, \hat{P}, \hat{\Sigma})$, such that

$$C_1 \|v\|_{L^q(T)} \leq \|\{v_i\}\|_{q,T} \leq C_2 \|v\|_{L^q(T)} \qquad \forall v \in P,$$

where

$$\|\{v_i\}\|_{q,T} = \begin{cases} \left(|T| \sum_{i=1}^d |v(z_i)|^q\right)^{1/q} & \text{if } 1 \leq q < \infty, \\ \max\limits_{1 \leq i \leq d} |v(z_i)| & \text{if } q = \infty, \end{cases}$$

and $|T|$ denotes the volume of T.

3. Prove that not any two Morley finite elements are affine equivalent.

4. Prove that the shape regularity condition of a family of two-dimensional finite element triangulations $\{\mathfrak{T}_h\}$ of a polygonal domain $\Omega \subset \mathbb{R}^2$ is equivalent to Zlámal's *maximum angle condition*: there exists a constant $\theta_0 > 0$ such that $\theta \geq \theta_0$ for any angle θ of any triangular element $T \in \cup_h \mathfrak{T}_h$.

5. Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a polygonal domain. Let $\{\mathfrak{T}_h\}$ be a family of regular finite element meshes of Ω parameterized by the mesh size h and $\{V_h\}$ a corresponding family of affine finite element spaces with each $V_h \subset C(\bar{\Omega})$. Show for each $p \in [1, \infty]$ that there exists a constant $C > 0$, independent of h, such that

$$\|v_h\|_{L^p(\partial\Omega)} \leq C h^{(1-n)/p} \|v_h\|_{L^p(\Omega)} \qquad \forall v_h \in V_h.$$

6. Let $n \geq 2$ be an integer and $\Omega \subset \mathbb{R}^n$ a Lipschitz polygonal domain. Let $\{\mathfrak{T}_h\}$ be a family of quasi-uniform finite element meshes parameterized by the mesh size h and $\{V_h\}$ a corresponding family of affine Lagrange finite element spaces with each $V_h \subset C(\bar{\Omega})$. For each h, let $\Pi_h : C(\bar{\Omega}) \to V_h$ denote the Lagrange finite element interpolation operator. Prove the following.
   (1) For each p with $1 \leq p \leq \infty$, there exists a constant $C > 0$, independent of h, such that

   $$\|\Pi_h u\|_{L^p(\Omega)} \leq C \|u\|_{L^p(\Omega)} \qquad \forall u \in C(\bar{\Omega}).$$

   (2) For each p with $n < p \leq \infty$, there exists a constant $C > 0$, independent of h, such that

   $$\|\Pi_h u\|_{W^{1,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \qquad \forall u \in W^{1,p}(\Omega).$$

7. Let $\{\mathfrak{T}_h\}$ be a regular family of finite element meshes, parameterized by the mesh size h, of a polygonal domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. Assume all the elements in $\cup_h \mathfrak{T}_h$ are affine equivalent to a single element $\hat{T} \subset \mathbb{R}^n$. For each $u \in L^1(\Omega)$ and each h, define $A_h u \in L^\infty(\Omega)$ to be a piecewise constant function with $A_h u = u_T$ on any $T \in \mathfrak{T}_h$, where

$$u_T = \frac{1}{|T|} \int_T u(x)\, dx \qquad \forall T \in \mathfrak{T}_h,$$

and $|T|$ denotes the volume of T. Let $1 \leq p \leq \infty$. Prove the following.

(1) If $u \in L^p(\Omega)$, then

$$\|\bar{u}_h\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}.$$

(2) If $u \in W^{1,p}(\Omega)$, then

$$\|u - \bar{u}_h\|_{L^p(\Omega)} \le C|u|_{W^{1,p}(\Omega)},$$

where $C > 0$ is constant independent of u and h.

8. (Clément type interpolation.) Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, $\{\mathcal{T}_h\}$ a family of regular finite element triangulations of Ω parameterized by the mesh size h, and $\{V_h\}$ the corresponding family of $H_0^1(\Omega)$-conforming, P_1 type, Lagrange finite element spaces. Fix h. Denote by $\mathcal{N}_h = \{z_1, \cdots, z_N\}$ and $\Phi_h = \{\phi_1, \cdots, \phi_N\}$, respectively, the set of all vertices of triangular elements of $\mathcal{T}_h$ and that of all the shape functions in V_h such that $\phi_i(z_j) = \delta_{ij}$ $(i, j = 1, \cdots, N)$. For each i $(1 \le i \le N)$, denote $\omega_i = \operatorname{supp} \phi_i$ and $P_i : L^2(\omega_i) \to P_1(\omega_i)$ the L^2 projection onto $P_1(\omega_i)$, i.e., for any $v \in L^2(\omega_i)$, $P_i v \in P_1(\omega_i)$ and $(v - P_i v, p)_{\omega_i} = 0$ for all $p \in P_1(\omega_i)$. Finally, define $\pi_h : L^2(\Omega) \to V_h$ by

$$\pi_h v = \sum_{i=1}^{N} P_i v(z_i) \phi_i \qquad v \in L^2(\Omega).$$

Denote by C a generic, positive constant that is independent of the diameter of any element in any of the meshes. Prove the following.

(1) If $T \in \mathcal{T}_h$ and $T \subseteq \omega_i$, then

$$\operatorname{diam} \omega_i \le C h_T, \qquad i = 1, \cdots, N,$$

where $h_T = \operatorname{diam} T$. Moreover,

$$\operatorname{card} \{T \in \mathcal{T}_h : T \subseteq \omega_i\} \le C. \qquad i = 1, \cdots, N.$$

(2) For any integers l and m with $0 \le m \le l \le 2$,

$$|v - P_i v|_{m,\omega_i} \le C\, (\operatorname{diam} \omega_i)^{l-m}\, |v|_{1,\omega_i} \qquad \forall v \in H^1(\omega_i),\ i = 1, \cdots, N.$$

(3) For $m = 0, 1$,

$$|p|_{m,\infty,T} \le C|T|^{-1/2} h_T^{-m} |p|_{0,T} \qquad \forall T \in \mathcal{T}_h,\ \forall p \in P_1(T),$$

and

$$|\phi_i|_{m,T} \le C|T|^{1/2} h_T^{-m} \qquad \forall T \in \mathcal{T}_h,\ i = 1, \cdots, N,$$

where $|T|$ denotes the area of T.

(4) For any $v \in L^2(\Omega)$,

$$|v - \pi_h v|_{0,\Omega} \le C|v|_{0,\Omega},$$
$$\lim_{h \to 0} |v - \pi_h v|_{0,\Omega} = 0.$$

For any $v \in H^1(\Omega)$ and $m = 0, 1$,

$$|v - \pi_h v|_{m,\Omega} \le C h^{1-m} |v|_{1,\Omega},$$
$$\lim_{h \to 0} |v - \pi_h v|_{1,\Omega} = 0.$$

For any $v \in H^2(\Omega)$ and $m = 0, 1$,

$$|v - \pi_h v|_{m,\Omega} \le C h^{2-m} |v|_{2,\Omega},$$

$$\left(\sum_{T \in \mathcal{T}_h} |v - \pi_h v|_{2,T}^2 \right)^{1/2} \le C |v|_{2,\Omega}.$$

(Hint: for $T = \Delta z_{i_1} z_{i_2} z_{i_3} \in \mathcal{T}_h$ and $v \in L^2(\Omega)$, show that

$$(\pi_h v - v)|_T = (P_{i_1} v - v)|_T + \sum_{j=2,3} \left[P_{i_j} v(z_{i_j}) - P_{i_1} v(z_{i_j}) \right] \phi_{i_j}|_T$$

and estimates the two terms on the right in suitable semi-norms.)

Chapter 4. Error Estimates for Elliptic Boundary Value Problems

1. Let V be a real Hilbert space with norm $\|\cdot\|$. Let $A : V \times V \to \mathbb{R}$ be a bilinear form satisfying
   (1) boundedness: there exists a constant $M > 0$ such that
   $$|A(v, w)| \le M \|v\| \, \|w\| \qquad \forall v, w \in V,$$
   (2) coercivity: there exists a constant $\gamma > 0$ such that
   $$A(v, v) \ge \gamma \|v\|^2 \qquad \forall v \in V.$$
   Let V_h a closed subspace of V. Let $u \in V$ and $u_h \in V_h$ satisfy
   $$A(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h.$$
   Prove for any $\xi_h \in V_h$ that
   $$\gamma \|u_h - \xi_h\| \le \sup_{v_h \in V_h,\, v_h \ne 0} \frac{A(u - \xi_h, v_h)}{\|v_h\|} \le M \|u_h - \xi_h\|.$$

2. (Optimality of finite element error estimates.) Fix an integer $n \ge 2$ and let $\mathcal{T}_h = \{[x_{i-1}, x_i] : i = 1, \cdots, n\}$ be a finite element mesh of $(0, 1)$, where $x_i = ih$ ($i = 0, \cdots, n$) and $h = 1/n$ is the mesh size. Let $V_h \subset H_0^1(0, 1)$ be the corresponding P_k type Lagrange finite element space for some integer $k \ge 1$. Let $u_h \in V_h$ be the unique finite element approximation of a function $u \in P_{k+1}(0, 1) \cap H_0^1(0, 1)$, defined by
   $$((u - u_h)', v_h') = 0 \qquad \forall v_h \in V_h,$$
   where $(\cdot, \cdot)$ denotes the $L^2(0, 1)$ inner product.
   (1) Prove that
   $$u_h\big(x_i^{(j)}\big) = u\big(x_i^{(j)}\big), \qquad j = 0, \cdots, k, \ i = 1, \cdots, n,$$
   where $\{x_i^{(j)}\}_{j=0}^k$ for each i ($1 \le i \le n$) are all the Lobatto points of order $k - 1$ on $[x_{i-1}, x_i]$, defined by
   $$x_i^{(j)} = \frac{x_{i-1} + x_i + h\xi^{(j)}}{2}, \qquad j = 0, \cdots, k,$$

where $\xi_0 = -1$, $\xi_k = 1$, and $\xi_1 < \cdots < \xi_{k-1}$ are the $k-1$ simple roots in $(-1, 1)$ of $L_k' \in P_{k-1}$ and

$$L_k(\xi) = \frac{1}{2^k k!} \frac{d^k}{d\xi^k} \left[\left(\xi^2 - 1\right)^{2k} \right]$$

is the kth Legendre polynomial.

(2) Prove that

$$\|u - u_h\|_{L^2(0,1)} = \frac{\left|u^{(k+1)}\right|}{(k+1)!} \left[\int_{-1}^{1} \prod_{j=0}^{k} \left(\xi - \xi^{(j)}\right)^2 d\xi \right]^{1/2} h^{k+1},$$

$$\|u' - u_h'\|_{L^2(0,1)} = \frac{\left|u^{(k+1)}\right|}{(k+1)!} \left\{ \int_{-1}^{1} \left[\frac{d}{d\xi} \prod_{j=0}^{k} \left(\xi - \xi^{(j)}\right) \right]^2 d\xi \right\}^{1/2} h^{k},$$

$$\|u - u_h\|_{L^\infty(0,1)} = \frac{\left|u^{(k+1)}\right|}{(k+1)!} \max_{-1 \le \xi \le 1} \left| \prod_{j=0}^{k} \left(\xi - \xi^{(j)}\right) \right| h^{k+1},$$

$$\|u' - u_h'\|_{L^\infty(0,1)} = \frac{\left|u^{(k+1)}\right|}{(k+1)!} \max_{-1 \le \xi \le 1} \left| \frac{d}{d\xi} \prod_{j=0}^{k} \left(\xi - \xi^{(j)}\right) \right| h^{k}.$$

3. Let $\Omega \subset \mathbb{R}^n$ be a convex polygonal domain, $\{\mathcal{T}_h\}$ a family of quasi-uniform finite element meshes parameterized by the mesh size h, and $\{V_h\}$ a corresponding family of affine, $H_0^1(\Omega)$-conforming, Lagrange finite element spaces with the reference finite element $(\hat{T}, \hat{P}, \hat{\Sigma})$ satisfying that $\hat{P} \supseteq P_k(\hat{P})$ for some integer $k \ge 1$. Let $u \in H_0^1(\Omega) \cap W^{k+1,\infty}(\Omega)$ and $u_h \in V_h$ for each h be the Galerkin finite element approximation of u defined by

$$(\nabla(u - u_h), \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

Prove that there exists a constant $C > 0$, independent of h and u, such that

$$\|u - u_h\|_{0,\infty,\Omega} \le C h^{k+1-n/2} \|u\|_{k+1,\infty,\Omega}.$$

4. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Recall that there exists $q_0 \in (2, \infty)$, depending only on Ω, that satisfies the following property: for any $q \in (1, q_0)$ and $f \in L^q(\Omega)$, there exists a unique $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

and

$$\|u\|_{2,q,\Omega} \le C \|f\|_{0,q,\Omega},$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product and $C > 0$ is a constant that only depends on Ω and q.

Let $\{\mathcal{T}_h\}$ be a family of quasi-uniform finite element meshes parameterized by the mesh size h and $\{V_h\}$ a corresponding family of affine, $H_0^1(\Omega)$-conforming, Lagrange finite element spaces. For each h, let $u_h \in V_h$ denote the Ritz-Galerkin finite element projection of a fixed $u \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$, defined by

$$(\nabla(u - u_h), \nabla v_h) = 0 \qquad \forall v_h \in V_h.$$

Show for each real number $p > q_0$ that there exists a constant $C > 0$, independent of u and h, such that

$$\|u - u_h\|_{0,p,\Omega} \leq Ch\|u - u_h\|_{1,p,\Omega}.$$

5. (Discrete maximum principle.) Consider the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with a Lipschitz continuous boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Let $\mathcal{T}_h$ be a finite element triangulation of Ω and $V_h \subset C(\bar{\Omega}) \cap H_0^1(\Omega)$ the corresponding piecewise linear finite element space. Let $\phi_1, \cdots, \phi_N$ be all the finite element shape functions defined by $\phi_i \in V_h$ $(i = 1, \cdots, N)$ and $\phi_i(z_j) = \delta_{ij}$ $(i, j = 1, \cdots, N)$, where $z_1, \cdots, z_N$ are all interior vertices of triangular elements in $\mathcal{T}_h$. Define $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ and $b = (b_i) \in \mathbb{R}^N$ by $a_{ij} = (\nabla\phi_i, \nabla\phi_j)$ and $b_i = (f, \phi_i)$, where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product. The finite element solution $u_h \in V_h$ of the given boundary value problem is determined by $u_h = \sum_{i=1}^N \xi_i\phi_i$, where $\xi = (\xi_i) \in \mathbb{R}^N$ is the unique solution of the linear system $A\xi = b$.

   Assume that all the angles of triangular elements in $\mathcal{T}_h$ are less than or equal to 90°. Prove the following.

   (1) All $a_{ii} > 0$ $(i = 1, \cdots, N)$, $a_{ij} \leq 0$ $(i, j = 1, \cdots, N, i \neq j)$, and $\sum_{j=1}^N a_{ij} \geq 0$ $(i = 1, \cdots, N)$. Moreover, there exists $i_0 \in \{1, \cdots, N\}$ such that $\sum_{j=1}^N a_{i_0 j} > 0$.

   (2) If all $b_i \leq 0$ $(i = 1, \cdots, N)$, then $u_h \leq 0$ on $\bar{\Omega}$.

6. Prove some properties for the special weight functions.