

**AMSC 698V: Advanced Topics in Applied Mathematics, Fall, 2003**  
**Mathematical and Computational Problems in Materials Science**  
**Instructor: Bo Li**

**HOMEWORK ASSIGNMENT 1**  
**(Due Wednesday, October 8, 2003)**

1. Let  $\varepsilon > 0$  and define  $I_\varepsilon : H^1(0, 1) \rightarrow \mathbb{R}$  by

$$I_\varepsilon[u] = \int_0^1 [u^2(u-1)^2 + \varepsilon u'^2] dx \quad \forall u \in H^1(0, 1).$$

Let  $0 < \lambda < 1$  and define

$$\mathcal{A}_\lambda = \left\{ u \in H^1(0, 1) : \int_0^1 u dx = \lambda \right\}.$$

Prove that there exists  $u \in \mathcal{A}_\lambda$  such that

$$I_\varepsilon[u] = \min_{v \in \mathcal{A}_\lambda} I_\varepsilon[v].$$

2. Define

$$I[u] = \int_0^1 \sqrt{u^2 + u'^2} dx \quad \forall u \in W^{1,1}(0, 1).$$

- (1) Prove that  $I : W^{1,1}(0, 1) \rightarrow \mathbb{R}$  is sequentially weakly lower semi-continuous.  
(2) Let

$$\mathcal{A} = \{u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = 1\}.$$

Prove that

$$I[u] > 1 \quad \forall u \in \mathcal{A}$$

and that

$$\inf_{u \in \mathcal{A}} I[u] = 1.$$

3. Let  $n \geq 1$  be an integer and  $\Omega \subset \mathbb{R}^n$  a bounded Lipschitz domain.

- (1) Let  $a, b \in \mathbb{R}^n$  and  $\lambda \in (0, 1)$ . Define  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$u(x) = b \cdot x + \int_0^{x \cdot (a-b)} \chi_\lambda(s) ds \quad \forall x \in \mathbb{R}^n,$$

where  $\chi_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  is the 1-periodic function such that  $\chi_\lambda(s) = 1$  for  $0 \leq s < \lambda$  and  $\chi_\lambda(s) = 0$  for  $\lambda \leq s < 1$ . For each integer  $k \geq 1$ , define  $u_k : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $u_k(x) = u(kx)/k$  for  $x \in \mathbb{R}^n$ . Finally, define  $\bar{u} : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\bar{u}(x) = [\lambda a + (1 - \lambda)b] \cdot x \quad \forall x \in \mathbb{R}^n.$$

Prove that

$$u_k \xrightarrow{*} \bar{u} \quad \text{in } W^{1,\infty}(\Omega).$$

- (2) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function and define

$$I[v] = \int_\Omega F(\nabla v(x)) dx \quad \forall v \in W^{1,\infty}(\Omega).$$

Assume that  $I : W^{1,\infty}(\Omega) \rightarrow \mathbb{R}$  is sequentially weakly-\* lower semi-continuous. Use the result of Part (1) to prove that  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex.

4. Let  $m \geq 1$  and  $n \geq 1$  be two integers. Let  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  be a smooth function. Prove that the following statements are equivalent:

(1) The function  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  is rank-one convex, i.e.,

$$F(\lambda A + (1 - \lambda)B) \leq \lambda F(A) + (1 - \lambda)F(B) \\ \forall A, B \in \mathbb{R}^{m \times n} \text{ with } \text{rank}(A - B) \leq 1, \forall \lambda \in (0, 1);$$

(2) For any  $A \in \mathbb{R}^{m \times n}$ ,  $\xi \in \mathbb{R}^m$ , and  $\eta \in \mathbb{R}^n$ , the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\varphi(t) = F(A + t\xi \otimes \eta) \quad \forall t \in \mathbb{R}$$

is convex;

(3) The function  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  satisfies the Legendre-Hadamard condition

$$(\xi \otimes \eta)^T \nabla^2 F(A) (\xi \otimes \eta) \geq 0 \quad \forall A \in \mathbb{R}^{m \times n} \quad \forall \xi \in \mathbb{R}^m \quad \forall \eta \in \mathbb{R}^n.$$

5. Let  $m \geq 1$  and  $n \geq 1$  be two integers and  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  a continuous function. Prove that  $F$  is quasiconvex if and only if

$$F(A) \leq \int_{(0,1)^n} F(A + \nabla v(x)) dx$$

for any  $v \in C^\infty(\mathbb{R}^n; \mathbb{R}^m)$  that is  $[0, 1]^n$ -periodic.

6. Let  $m \geq 1$  and  $n \geq 1$  be two integers and  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  a continuous function. Prove that, if  $F$  is quasiconvex, then it is rank-one convex.

7. Let  $n \geq 1$  be an integer and  $\Omega \subset \mathbb{R}^n$  a bounded Lipschitz domain. Assume that

$$u_k \xrightarrow{*} u \quad \text{in } W^{1,\infty}(\Omega; \mathbb{R}^n).$$

Prove that

$$\det \nabla u_k \xrightarrow{*} \det \nabla u \quad \text{in } L^\infty(\Omega).$$

8. Let  $m \geq 1$  and  $n \geq 1$  be two integers and  $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  a continuous function. Prove that, if  $F$  is polyconvex, then it is quasiconvex.