AMSC 698V: Advanced Topics in Applied Mathematics, Fall, 2003
Mathematical and Computational Problems in Materials Science
Instructor: Bo Li

HOMEWORK ASSIGNMENT 2
(Due Monday, November 3, 2003)

1. Let $A, B \in \mathbb{R}^{3\times 3}$, $n \in \mathbb{R}^3$ with $|n| = 1$, and $\alpha \in \mathbb{R}$. Prove that there exists a continuous mapping $y : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\nabla y(x) = \begin{cases} A & \text{if } x \cdot n < \alpha, \\ B & \text{if } x \cdot n > \alpha, \end{cases}$$

if and only if

$$A - B = a \otimes n$$

for some $a \in \mathbb{R}^3$.

2. Let $U_1 = \operatorname{diag}\{\alpha, \beta, \gamma\}$ and $U_2 = \operatorname{diag}\{\beta, \alpha, \gamma\}$ with α, β, and γ positive real numbers and $\alpha \neq \beta$. Find all $(Q, a, n) \in (\mathrm{SO}\,(3), \mathbb{R}^3, \mathbb{R}^3)$ with $a \neq 0$ and $|n| = 1$ such that

$$QU_1 - U_2 = a \otimes n.$$

Show for any $\lambda \in (0, 1)$ that

$$F_\lambda := \lambda Q U_1 + (1 - \lambda) U_2 \notin \mathrm{SO}\,(3) U_1 \cup \mathrm{SO}\,(3) U_2.$$

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\{\nu_x\}_{x\in\Omega}$ be the family of Young measures associated with a bounded sequence $\{u_k\}$ in $L^\infty(\Omega; \mathbb{R}^m)$. Let $1 < p < \infty$. Prove that $u_k \to u$ in $L^p(\Omega; \mathbb{R}^m)$ if and only if $\nu_x = \delta_{u(x)}$ for a.e. $x \in \Omega$.

4. Let $I : W^{1,4}(0, 1) \to \mathbb{R}$ be defined by

$$I[u] = \int_0^1 \left[\left(u'^2 - 1 \right)^2 + u^2 \right] dx \qquad \forall u \in W^{1,4}(0, 1).$$

   (1) Prove that $\inf_{u\in W^{1,4,}(0,1)} I[u] = 0$ but $I[u] > 0$ for any $u \in W^{1,4}(0, 1)$.

   (2) Let $u_k \in W^{1,4}(0, 1)$ $(k = 1, \ldots)$ be such that $\lim_{k\to\infty} I[u_k] = 0$. Let $\{\nu_x\}_{x\in(0,1)}$ be the family of Young measures associated with $\{u_k'\}$. Prove that

$$\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \qquad \text{a.e. } x \in (0, 1).$$

5. Let $\Omega = (0, 1) \times (0, 1)$. Define $I : W_0^{1,4}(\Omega) \to \mathbb{R}$ by

$$I[u] = \int_\Omega \left[\left((\partial_{x_1} u)^2 - 1 \right)^2 + (\partial_{x_2} u)^2 \right] dx \qquad \forall u \in W_0^{1,4}(\Omega).$$

   (1) Prove that $\inf_{u\in W_0^{1,4}(\Omega)} I[u] = 0$ but $I[u] > 0$ for any $u \in W_0^{1,4}(\Omega)$.

   (2) Let $u_k \in W_0^{1,4}(\Omega)$ $(k = 1, \ldots)$ be such that $\lim_{k\to\infty} I[u_k] = 0$. Let $\{\nu_x\}_{x\in\Omega}$ be the family of Young measures associated with $\{\nabla u_k\}$. Prove that

$$\nu_x = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(-1,0)} \qquad \text{a.e. } x \in \Omega.$$

6. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\lambda \in (0,1)$, and $A, B \in \mathbb{R}^3$ with

$$A - B = a \otimes n$$

for some nonzero $a, n \in \mathbb{R}^3$. Let

$$y_0(x) = (\lambda A + (1-\lambda)B)x \qquad \forall x \in \Omega.$$

Let $\{y_k\}$ be a bounded sequence in $W^{1,\infty}(\Omega;\mathbb{R}^3)$ such that $y_k(x) = y_0(x)$ for all $x \in \partial\Omega$ and that the corresponding sequence of gradients $\{\nabla y_k\}$ generates the Young measures

$$\nu_x = \lambda\delta_A + (1-\lambda)\delta_B \qquad \text{a.e. } x \in \Omega.$$

Prove that $y_k \overset{*}{\rightharpoonup} y_0$ in $W^{1,\infty}(\Omega;\mathbb{R}^3)$ and that $(\nabla y_k)w \to (\nabla y_0)w$ in $L^p(\Omega;\mathbb{R}^3)$ for any $w \in \mathbb{R}^3$ such that $w \cdot n = 0$ and any $p \in [1,\infty)$.