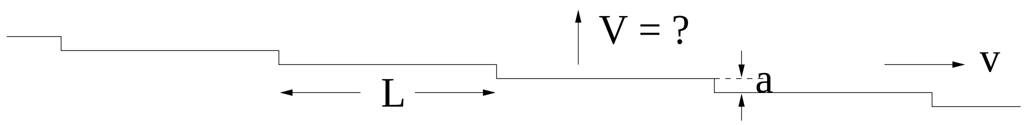
AMSC 698V: Advanced Topics in Applied Mathematics, Fall, 2003
Mathematical and Computational Problems in Materials Science
Instructor: Bo Li

HOMEWORK ASSIGNMENT 3
(Due Friday, December 12, 2003)

1. Assume a train of periodical steps, with a uniform step width L and mono-layer step height a, move horizontally at a constant speed v. Find the constant speed V at which the surface moves vertically.



2. Let $\Omega(t) \subset \mathbb{R}^d$ be a smooth domain depending on time t smoothly. Let $\rho(x,t)$ be a smooth function defined for all $x \in \Omega(t)$ for all time $t \in [0,T]$ for a given constant $T > 0$. Show that

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\,dx = \int_{\Omega(t)} \partial_t \rho(x,t)\,dx + \int_{\Gamma(t)} v_n(x,t)\rho(x,t)\,dS,$$

for $t \in (0,T)$, where $\Gamma(t)$ is the boundary of $\Omega(t)$ and v_n the normal velocity of $\Gamma(t)$.

3. Let $\Gamma(t) : y = Y(x,t)$ be a curve moving in a diffusional field with concentration $u = u(x,y,t)$ governed by the quasi-steady system

$$\begin{aligned}
&\Delta u = 0 \qquad \text{in } \Omega_+(t) \cup \Omega_-(t),\\
&u = \kappa \qquad \text{on } \Gamma(t),\\
&\partial_y u(x,y) \to F_\pm \qquad \text{as } y \to \pm\infty,\\
&v_n = [\nabla u \cdot n],
\end{aligned}$$

where

$$\begin{aligned}
\Omega_+(t) &= \{(x,y) \in \mathbb{R}^2 : y > Y(x,t)\},\\
\Omega_-(t) &= \{(x,y) \in \mathbb{R}^2 : y < Y(x,t)\},
\end{aligned}$$

κ is the curvature of $\Gamma(t)$ defined by

$$\kappa(x,t) = \frac{Y_{xx}}{(1+Y_x^2)^{3/2}},$$

both F_+ and F_- are two constants, n is the unit normal of $\Gamma(t)$ pointing from $\Omega_-(t)$ to $\Omega_+(t)$, and the brackets denote the jump across the interface $\Gamma(t)$ defined by $[w] = w|_{\Omega_+(t)} - w|_{\Omega_-(t)}$.

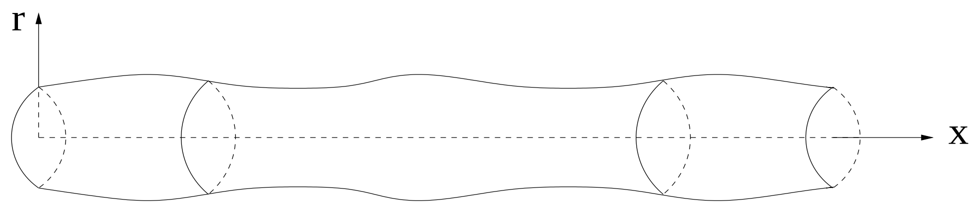
Assume that $Y(x,t)$ and $u(x,y,t)$ are both periodic in x. Perform the linear stability analysis on this system to obtain the growth rate for a sinusoidal perturbation with wavenumber k,

$$\omega(k) = (F_+ + F_-)k - 2k^3.$$

4. The motion by surface diffusion of an axially symmetric cylindrical body is governed by the fourth-order nonlinear diffusion equation for the radius $r = r(x, t)$:

$$r_t = \frac{1}{r}\left\{ \frac{r}{[1+(r_x)^2]^{1/2}} \left(\frac{1}{r\,[1+(r_x)^2]^{1/2}} - \frac{r_{xx}}{[1+(r_x)^2]^{3/2}} \right)_x \right\}_x,$$

where the subscript t or x denotes a corresponding partial derivative. Show that a cylinder with a constant radius is linearly stable with respect to such a motion if and only the wavelength of the perturbation is smaller than the circumference of the unperturbed cylinder.



5. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function that divides the plane $\mathbb{R}^2$ into a bounded open set $\Omega_- = \{x \in \mathbb{R}^2 : \phi(x) < 0\}$, an unbounded open set $\Omega_+ = \{x \in \mathbb{R}^2 : \phi(x) > 0\}$, and the curve $\Gamma = \{x \in \mathbb{R}^2 : \phi(x) = 0\}$. Let n be the unit normal of Γ pointing from Ω_- to Ω_+ and κ the curvature of Γ with the convention that it is positive if the curve is a circle. Show that

$$n = \frac{\nabla \phi}{|\nabla \phi|} \qquad \text{and} \qquad \kappa = \nabla \cdot n = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}.$$

6. Write a code to implement the level-set method with the finite difference discretization to solve the following problem:

$$\begin{aligned} &\partial_t u = \Delta u \qquad \text{in } \Omega \times (0, T), \\ &u : \Omega\text{-periodic for all } t \in [0, T], \\ &u = -\kappa \qquad \text{on } \Gamma(t), \\ &v_n = -[\nabla u \cdot n] \qquad \text{on } \Gamma(t), \end{aligned}$$

where $\Omega = (0, L_1) \times (0, L_2)$, $T > 0$, the initial curve $\Gamma(0)$, and the initial concentration $u(\cdot, 0) : \Omega \to \mathbb{R}$ are given, n and κ are the unit normal and curvature of the interface $\Gamma(t)$, respectively, and the brackets denote the jump across the interface $\Gamma(t)$. Suppose the interface $\Gamma(t)$ divides Ω into $\Omega_s(t)$ and $\Omega_l(t)$. Then, the normal n always points from $\Omega_s(t)$ to $\Omega_l(t)$, the curvature is positive if $\Gamma(t)$ is a circle, and the jump is defined by $[w] = w|_{\Omega_s(t)} - w|_{\Omega_l(t)}$.