$$\Lambda = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}, \quad \sqrt{\Lambda} = \begin{bmatrix} \sqrt{\sigma_1} & & \\ & \ddots & \\ & & \sqrt{\sigma_n} \end{bmatrix}. \quad \text{Then}$$

$$Rq_i = Q\sqrt{\Lambda}\,Q^T q_i = Q\sqrt{\Lambda}\,e_i = Q\sqrt{\sigma_i}\,e_i = \sqrt{\sigma_i}\,Qe_i = \sqrt{\sigma_i}\,q_i$$

This is what I meant in Problem 4 above, that multiplying the identity out, we see the eigenvectors of A are precisely the eigenvectors of $\sqrt{A}$, and its corresponding eigenvalue is $\sqrt{\lambda}$. By above if $\lambda > 0$, of course $\sqrt{\lambda} > 0$, hence R is PSD. The last part is a routine verification.

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4. Decide whether the following matrices are positive definite, negative definite, semidefinite, or indefinite.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \Rightarrow A \text{ is indefinite.}$$

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5/3 \end{bmatrix} \Rightarrow B \text{ is PSD.}$$

C is NSD: Pivots of $C = -B$ are the negative pivots of B.

$D = A^{-1}$ is indefinite. Signs of eigenvalues of A are $(+,+,-)$. Recall eigenvalues of A^{-1} are λ^{-1}. Then eigenvalues of A^{-1} have same sign as eigenvalues of A.

Consider $-x^2 - 5y^2 - 9z^2 - 4xy - 6xz - 8yz = 1$. This is the quadratic form associated to A. So this question is $-(x^TAx) = 1$, or $x^TAx = -1$. There is a solution to this: Notice A is indefinite. Hence it is not the case $x^TAx \geq 0 \ \forall x$. ie $\exists x$ s.t. $x^TAx < 0$.