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Math 102 Winter 2017.

B.C. 2/7/2017

Midterm Rev.

$$\text{Ch 1. #3. } A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\#4. \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} B^{-1}.$$

$$\rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 4 & 1 \\ 0 & 0 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R.$$

Try again:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{bmatrix}$$

$$\text{check: } LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 15/2 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8/3 & 0 \\ 0 & 0 & 5/2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \quad = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = A$$

$$A = LDV$$

$$\text{So, } A = LU$$

[2]

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad A = LU$$

$$A\vec{x} = \vec{b}, \quad LU\vec{x} = \vec{b}. \quad \text{Let } \vec{y} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = U\vec{x}.$$

$$L\vec{y} = \vec{b} \quad \begin{cases} u = 2 \\ v = 2 \\ 3u + w = 5 \end{cases} \Rightarrow \vec{y} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$U\vec{x} = \vec{y} \quad \begin{cases} 2x + 3y + 3z = 2 \\ 5y + 7z = 2 \\ -z = -1 \end{cases} \quad \begin{aligned} z &= 1 \\ y &= -1 \\ x &= [2 - 3(-1) - 3(1)]/2 \\ &= 1 \end{aligned}$$

$$\text{So, } \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Ch 2. #1. $\{\text{polynomials of deg. 2}\}$ is not a subspace.

$p(x) = x^2, q(x) = -x^2 + 1$. both have deg. 2

But, $p(x) + q(x) = 1$: deg 0.

$\{A \in M_{2 \times 2} : \det A = 0\}$ is not a subspace
of $M_{2 \times 2}$ (the space of all 2×2 matrices).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \det A = 0, \quad \det B = 0$$

But $\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$. $A+B$ is not in this set.

$$\begin{aligned} N(A) &= \{\vec{x} : A\vec{x} = \vec{0}\}. \quad A\vec{x} = \vec{0} \Rightarrow A(c\vec{x}) = cA\vec{x} = \vec{0} \\ &\quad c \in \mathbb{R} \Rightarrow c\vec{x} \in N(A) \\ &\quad \vec{x} - \vec{y} \in N(A) \Rightarrow A\vec{x} = \vec{0}, A\vec{y} = \vec{0} \\ &\quad \Rightarrow A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0} \\ &\quad \Rightarrow \vec{x} + \vec{y} \in N(A). \end{aligned}$$

The zero matrix.

$$\#2. \quad aU_1 + U_3 = 3U_1 + bU_2 + cU_3$$

$$\Rightarrow (a-3)U_1 - bU_2 + (1-c)U_3 = 0$$

$$U_1, U_2, U_3 \in L.I. \Rightarrow a-3=0, -b=0, 1-c=0 \quad \text{So, } a=3, b=0, c=1.$$

Why $n+1$ vectors in \mathbb{R}^n are always L.D.?

Let $\vec{a}_1, \dots, \vec{a}_{n+1} \in \mathbb{R}^n$. Suppose $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_{n+1}\vec{a}_{n+1} = \vec{0}$

$$\underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_{n+1} \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vdots \\ \vec{0} \end{bmatrix}$$

$A \quad \vec{x} \quad = \quad \vec{0}$
 $n \times (n+1) \quad (n+1) \times 1 \quad n \times 1$

fewer eqns
more unknowns

$$\text{Rank}(A) = \# \text{ pivots} \leq n.$$

$$\# \text{ free variables} = n+1 - \# \text{ pivots} \geq 1$$

So, at least 1 free variable. So, at least one non-zero sol'n \vec{x} . So, $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{n+1}$ are L.D.

$\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ is a L.C. of $\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$ means

there are numbers x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}}_{\text{same system as in \#4. ch 1. (of this review).}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So, yes!

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 7 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

\Rightarrow non zero sol'n \vec{x} exist. So, L.D.

[4]

$$\#3. A\vec{x} = \vec{0} \quad A = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_1 = 5x_3 \\ x_2 = -x_3$$

$$\vec{x} = x_3 \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \quad x_3: \text{free variable (any number)}$$

$$A\vec{x} = \vec{b} \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} & & x_1 - 5x_3 = 1 & \Rightarrow x_1 = 1 + 5x_3 \\ & & x_2 + x_3 = 4 & \Rightarrow x_2 = 4 - x_3 \end{array} \right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + 5x_3 \\ 4 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \quad x_3: \text{free variable.}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 \end{array} \right]$$

condition. $b_3 - b_2 - 2b_1 = 0$

In general, $\text{Rank}(A+B) \neq \text{Rank}(A) + \text{Rank}(B)$.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In general, $\text{Rank}(A+B) \neq \text{Rank}A + \text{Rank}B$.

For example, $-A = B = I$, 2×2 . $A = -I$, $B = I$, $A+B=0$.

#4. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ are L.I. $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{Row ops}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -4 \end{bmatrix}$

Add $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are L.I.

Dimension of {polynomials of deg. ≤ 4 } = 5

A basis, $1, x, x^2, x^3, x^4$.

$$M_{3 \times 4}: \text{dim} = 2. \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let E_{ij} = the 3×4 matrix with (i,j) -entry = 1, others 0
 $i=1, 2, 3$, $j=1, 2, 3, 4$. Then $\{E_{11}, E_{12}, \dots, E_{34}\}$ is a basis.

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dimension of { 5×5 symmetric matrices}

$$= 5 + 4 + 3 + 2 + 1 = 15$$

#5. $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \\ 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $C(A)$: $\left[\begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right], \left[\begin{array}{c} 3 \\ 9 \\ 3 \end{array} \right]$

Warning:
not $\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 3 \\ 0 \\ 0 \end{array} \right]$

pivot columns
 $\begin{array}{c} 1 \\ 9 \\ 9 \end{array}$

In fact, \vec{a}_1, \vec{a}_3 : L.I. $\vec{a}_2 \parallel \vec{a}_1$, $\vec{a}_4 = \vec{a}_3 - \vec{a}_1$.

$$A\vec{x} = \vec{0} \quad x_1 + 3x_2 - x_4 = 0 \quad x_1 = -3x_2 + x_4$$

$$x_3 + x_4 = 0 \quad x_3 = -x_4$$

$$\vec{x} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

A basis for $N(A)$ is $\left[\begin{array}{c} -3 \\ 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \\ -1 \\ 1 \end{array} \right]$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

① $C(A) = \text{span} \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \right\}$ - the xy -plane

② $N(A)$. $A\vec{x} = \vec{0} \quad x_2 = 0 \quad x_3 = 0$.

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (\text{the } x\text{-axis}).$$

③ Now replace $C(A^T)$. $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$C(A^T)$ is spanned

by $\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$ yz -plane.

④ left null space of A . $N(A^T)$ $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ x_3 : free variable
the z -axis.

$$B = \begin{bmatrix} 2 & 1 & 1 \\ -4 & -2 & -2 \\ 8 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{yes, rank 1.}$$

$$B = \vec{a} \vec{b}^T \quad \vec{a} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

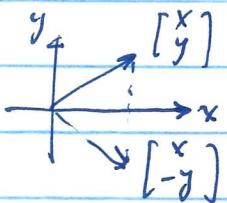
How to get \vec{a} and \vec{b} ?

$$B = \begin{bmatrix} \vec{b}^T \\ -2\vec{b}^T \\ 4\vec{b}^T \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \vec{b}^T$$

Let $\vec{b}^T = \text{row 1}$.

so, this is \vec{a} . $\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$.

#6. 2×2 matrix: reflection about the x-axis



The reflection transforms $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ -y \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{let } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_1) = \vec{e}_1 = 1 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2$$

$$T(\vec{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \cdot \vec{e}_1 + (-1) \vec{e}_2$$

$$\text{The matrix is } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad \text{check: } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Rotation counterclockwise by α : $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

$$\alpha = \frac{\pi}{4}, \quad \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

let $P_2 = \{\text{all polynomials of deg} \leq 2\}$.

Basis $e_0(x) = 1, e_1(x) = x, e_2(x) = x^2$.

$$T = \frac{d}{dx} \quad T(e_0) = 0 = 0 \cdot e_0 + 0 \cdot e_1 + 0 \cdot e_2$$

$$T(e_1) = \frac{d}{dx}(x) = 1 = 1 \cdot e_0 + 0 \cdot e_1 + 0 \cdot e_2$$

$$T(e_2) = \frac{d}{dx}(x^2) = 2x = 0 \cdot e_0 + 2e_1 + 0 \cdot e_2$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Ch 3 #1. } \|\vec{a}\| = \sqrt{9+1+1} = \sqrt{11} \quad \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = -3-1+2 = -2$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{-2}{\sqrt{11} \sqrt{6}} = -\frac{2}{\sqrt{66}}.$$

\vec{a}, \vec{b} not orthogonal.

$$\text{Also, } \vec{v}^T \vec{u} = \vec{u}^T \vec{v} = 0$$

$\vec{u} \perp \vec{v}$ means $\vec{u}^T \vec{v} = 0$. In this case

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v})^T (\vec{u} - \vec{v}) = (\vec{u}^T - \vec{v}^T)(\vec{u} - \vec{v})$$

$$= \vec{u}^T (\vec{u} - \vec{v}) - \vec{v}^T (\vec{u} - \vec{v}) = \vec{u}^T \vec{u} - \vec{u}^T \vec{v} - \vec{v}^T \vec{u} + \vec{v}^T \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

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$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \iff \|\vec{x} + \vec{y}\|^2 \leq (\|\vec{x}\| + \|\vec{y}\|)^2$$

$$\iff \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\vec{x}^T \vec{y} \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$$

$\iff \vec{x}^T \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$. True by the Cauchy-Schwarz inequality.

#2. "V and W are orthogonal to each other" means any $\vec{v} \in V$ and any $\vec{w} \in W$ are orthogonal $\vec{v} \perp \vec{w}$.

But, V and W may not be orthogonal complement.

For example, in \mathbb{R}^3 , $V = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ (the x-axis) and $W = \left\{ \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} : b \in \mathbb{R} \right\}$ (the y-axis).

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \text{ Find all } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ s.t. } \vec{x} \perp \vec{a}, \vec{x} \perp \vec{b}.$$

$$\begin{aligned} \vec{x}^T \vec{a} = 0 &\iff x_1 + x_3 = 0 & x_1 + x_3 = 0 \\ \vec{x}^T \vec{b} = 0 &\iff -x_1 + x_2 + 2x_3 = 0 & x_2 + 3x_3 = 0 \\ \vec{x} = \begin{bmatrix} -x_3 \\ -3x_3 \\ x_3 \end{bmatrix} &= x_3 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}. & x_3: \text{free variable.} \end{aligned}$$

$$\text{So, } (\text{span}\{\vec{a}, \vec{b}\})^\perp = \text{span}\left\{\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\}.$$

$$\#3. \vec{a} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \text{ Proj. of } \vec{b} \text{ onto } \vec{a}:$$

$$\begin{aligned} P_{\vec{a}} \vec{b} &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a} = \frac{-3 - 1 + 2}{11} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = -\frac{2}{11} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{2}{11} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

$$\text{Proj. matrix: } P_{\vec{a}} = \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} = \frac{1}{11} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 & -3 & 3 \\ -3 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\#4. \quad A^T A \hat{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \quad \begin{array}{l} x_1 + 2x_2 = 1 \\ 3x_2 = 1 \end{array} \quad \begin{array}{l} x_1 = 1 - 2 \cdot (\frac{1}{3}) = \frac{1}{3} \\ x_2 = \frac{1}{3} \end{array}$$

$$\text{So, } \hat{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}. \quad \vec{p} = A \hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$P = A (A^T A)^{-1} A^T$$

$$[A^T A \quad I] = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} + \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\underbrace{\frac{1}{3}}_{A^T A} & \frac{2}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad (A^T A)^{-1}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$