
Midterm 2 Review

1. The solution to the initial-value problem of diffusion equation \( u_t - ku_{xx} = 0 \) \((-\infty < x < \infty, t > 0)\) with \( u(x, 0) = \phi(x) \) \((-\infty < x < \infty)\) is given by

\[
u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, dy \quad (-\infty < x < \infty, t > 0),\]

What is the formula of \( S(x, t) \)? We need this to solve the half-line diffusion problem.

2. Given a function \( \phi = \phi(x) \) defined on \( x > 0 \). How to extend it to be an odd (or even) function on the entire line \(-\infty < x < \infty)\?

3. Consider the diffusion equation \( u_t - ku_{xx} = 0 \) on the half line \( x > 0 \) with the initial condition \( u(x, 0) = \phi(x) \) \((x > 0)\) and some boundary condition at \( x = 0 \). Find the solution if the boundary condition is: (1) \( u(0, t) = 0 \) \((t > 0)\); or (2) \( u_x(0, t) = 0 \) \((t > 0)\).

4. The solution to the initial-value problem of the wave equation \( u_{tt} - c^2 u_{xx} = 0 \) \((-\infty < x < \infty, t > 0)\) with \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) \((-\infty < x < \infty)\) is given by d’Alembert’s formula. What is that formula? What is the domain of dependence? We need this formula for solving the half-line wave equation.

5. Consider the wave equation \( u_{tt} - c^2 u_{xx} = 0 \) on the half line \( x > 0 \) with the initial condition \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) \((x > 0)\) and some boundary condition at \( x = 0 \). Find the solution if the boundary condition is: (1) \( u(0, t) = 0 \) \((t > 0)\); or (2) \( u_x(0, t) = 0 \) \((t > 0)\).

6. Consider the wave equation \( u_{tt} - c^2 u_{xx} = 0 \) on a finite interval \((0, l)\) with the initial condition \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) \((0 < x < l)\) and the homogeneous Dirichlet or Neumann boundary conditions. Obtain the solution formula using the method of reflection, and find the solution \( u = u(x, t) \) at a given point \((x, t)\) using the geometrical construction of reflection.

7. Solve the diffusion equation \( u_t - ku_{xx} = f(x, t) \) \((-\infty < x < \infty, t > 0)\) with the initial condition \( u(x, 0) = \phi(x) \) \((-\infty < x < \infty)\).

8. How to solve the half-line diffusion equation \( u_t - ku_{xx} = f(x, t) \) \((x > 0, t > 0)\) with the initial condition \( u(x, 0) = \phi(x) \) \((x > 0)\) with the boundary condition: (1) \( u(0, t) = g(t) \) \((t > 0)\); or (2) \( u_x(0, t) = h(t) \) \((t > 0)\)?

9. Solve the wave equation \( u_{tt} - c^2 u_{xx} = f(x, t) \) on the entire line with the initial condition \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \).

10. Solve the wave equation \( u_{tt} - c^2 u_{xx} = f(x, t) \) on the half-line with the initial condition \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) and the boundary condition: (1) \( u(0, t) = g(t) \) \((t > 0)\); or (2) \( u_x(0, t) = h(t) \) \((t > 0)\).

11. What are the eigenvalues and eigenfunctions for \(-X'' = \lambda X \) \((0 < x < l)\) with the boundary conditions: (1) \( X(0) = X(l) = 0 \)? and (2) \( X'(0) = X'(l) = 0 \)?

12. How to solve the initial-boundary-value problem of the diffusion equation \( u_t - ku_{xx} = 0 \) \((0 < x < l, t > 0)\) with the initial condition \( u(x, 0) = \phi(x) \) and the boundary conditions: (1) \( u(0, t) = 0 \) and \( u(l, t) = 0 \) \((t > 0)\); and (2) \( u_x(0, t) = 0 \) and \( u_x(l, t) = 0 \) \((t > 0)\)?

13. Use the method of separation of variables to solve the wave equation \( u_{tt} - c^2 u_{xx} = 0 \) on a finite interval with the initial conditions \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) and the boundary conditions (1) \( u(0, t) = 0 \) and \( u(l, t) = 0 \) \((t > 0)\); or (2) \( u_x(0, t) = 0 \) and \( u_x(l, t) = 0 \) \((t > 0)\).