Math 110A: Introduction to Partial Differential Equations, Winter 2014

Reivew for Final Exam

Part 1. Concept. First-Order Equations. Classification of Second-Order Equations.

1. Basic concept of PDE. Order of a PDE. Linear or nonlinear PDE. Initial conditions. Three types of boundary conditions: Dirichlet, Neumann, and Robin. Also: periodic boundary conditions. Well-posed problems.
2. Find the general solution to a simple equation, e.g., $u_{xx} = 0$ or $u_{xy} = 0$, by integration.
3. First-order linear equations of two variables: $a(x, y)u_x + b(x, y)u_y = f(x, y)$, where a, b, and f are given functions. Sometimes, initial conditions are given. Such an equation can be solved by the method of characteristics (cf. Section 1.2). Examples: (1) Find the general solution to $4u_x - 7u_y = 0$; (2) Find the general solution to $u_x + 2xy^2u_y = 0$; and (3) Solve $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.
4. Classification of second-order linear equations. Theorem 1 on page 28. Example: What is the type of the equation $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$, elliptic, parabolic, or hyperbolic? How to reduce (or transform) it to a form without the mixed derivative?

Part 2. Wave Equation

1. D'Alembert's formula for solution to $u_{tt} - c^2u_{xx} = 0$ ($x \in \mathbb{R}, t > 0$), and $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ ($x \in \mathbb{R}$). Domain of dependence and domain of influence.
2. Solve $u_{tt} - c^2u_{xx} = f(x, t)$ (with a source term) for $x \in \mathbb{R}$ and $t > 0$ with $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$. A technical point: how to integrate a function on a triangular region?
3. Method of reflection for solving $u_{tt} - c^2u_{xx} = 0$ ($x > 0, t > 0$) with $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ ($x > 0$) and some boundary condition at $x = 0$. Find the solution if the boundary condition is: (1) $u(0, t) = 0$ ($t > 0$); or (2) $u_x(0, t) = 0$ ($t > 0$). How this method works for inhomogeneous equation $u_{tt} - c^2u_{xx} = f(x, t)$ on a half-line $x > 0$ with inhomogeneous boundary conditions $u(0, t) = g(t)$ or $u_x(0, t) = h(t)$?
4. Method of reflection for $u_{tt} - c^2u_{xx} = 0$ on a finite interval $(0, l)$ with $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ ($0 < x < l$) and the homogeneous Dirichlet or Neumann boundary conditions. Find the solution $u = u(x, t)$ at a given point (x, t).
5. Use the energy method to prove the energy conservation ($E'(t) = 0$) or dissipation ($E'(t) \le 0$). Use this method to prove the uniqueness of solution to an initial-value or initial-boundary-value problem of wave equation.

Part 3. Diffusion Equation

1. Maximum principle. How to apply the Maximum Principle to find bounds of a solution to diffusion equation? Uniqueness and stability for initial-boundary-value problem for diffusion equation: statement and proofs.
2. Formula of solution to $u_t - ku_{xx} = 0$ ($-\infty < x < \infty, t > 0$) with $u(x, 0) = \phi(x)$ ($-\infty < x < \infty$). How the initial-condition is interpreted? Formula of the heat kernel or Gaussian kernel $S(x, t)$.
3. Solve the diffusion equation $u_t - c^2u_{xx} = f(x, t)$ (with a source term) on the entire line with the initial condition $u(x, 0) = \phi(x)$. See Eq. (1) and Eq. (2) on page 67.
4. Method of reflection for $u_t - ku_{xx} = 0$ on the half line $x > 0$ with the initial condition $u(x, 0) = \phi(x)$ ($x > 0$) and some boundary condition at $x = 0$. Find the solution if the boundary condition is: (1) $u(0, t) = 0$ ($t > 0$); or (2) $u_x(0, t) = 0$ ($t > 0$). How this method works for inhomogeneous equation $u_t - ku_{xx} = f(x, t)$ on a half-line $x > 0$ with

inhomogeneous boundary conditions $u(0,t) = g(t)$ or $u_x(0,t) = h(t)$?

Part 4. Separation of Variables. Eigenvalue and Eigenfunctions. Generalized Fourier Series.

1. Method of separation of variables for solving the wave equation $u_{tt} - c^2 u_{xx} = 0$ with the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, or the diffusion equation $u_t - k u_{xx} = 0$ with the initial conditions $u(x,0) = \phi(x)$, and the boundary conditions: (1) $u(0,t) = 0$ and $u(l,t) = 0$ $(t > 0)$; or (2) $u_x(0,t) = 0$ and $u_x(l,t) = 0$ $(t > 0)$; or (3) a combination of different boundary conditions; or (4) periodic boundary condition. The method can be applied to other similar equations.
2. Eigenvalues and eigenfunctions for $-X'' = \lambda X$ with the boundary conditions: (1) $X(0) = X(l) = 0$; (2) $X'(0) = X'(l) = 0$; and (3) $2l$-periodic boundary conditions. Orthogonality of eigenfunctions.
3. Determine coefficients of Fourier sine series expansion, Fourier cosine series expansion, and the full Fourier series expansion of a function in a respective interval. The complex form of Fourier series.
4. Convergence pointwise, uniformly, and in the mean-square sense. Convergence of Fourier sine series, Fourier cosine series, and the full Fourier series. Theorems 3 and 4 on pages 128 and 129.
5. Inhomogeneous boundary conditions and source term. The method of expansion in Section 5.6. For $u(0,t) = \alpha(t)$ and $u(l,t) = \beta(t)$ $(t > 0)$, define $v(x,t) = (1 - x/l)\alpha(t) + (x/l)\beta(t)$. Then $w(x,t) = u(x,t) - v(x,t)$ satisfies the homogeneous boundary condition $w(0,t) = 0$ and $w(l,t) = 0$ $(t > 0)$. For $u_x(0,t) = \alpha(t)$ and $u_x(l,t) = \beta(t)$ $(t > 0)$, one can define $v(x,t) = (x - x^2/2l)\alpha(t) + (x^2/2l)\beta(t)$. Then $w(x,t) = u(x,t) - v(x,t)$ satisfies the homogeneous boundary condition $w_x(0,t) = 0$ and $w_x(l,t) = 0$ $(t > 0)$.

Part 5. Harmonic Functions

1. What is a harmonic function? What is Laplace's equation? Boundary conditions? Maximum Principle for harmonic functions.
2. Separation of variables for boundary-value problem of Laplace's equation.
3. Laplace's equation in polar coordinates. Poisson's formula. Mean-Value Theorem. Find the value of $u(0,0)$ for a harmonic function $u = u(x,y)$ on the unit disk with the boundary-value $u(x,y) = 1 + 2\cos\theta$ $(0 \le \theta < 2\pi)$.