Math 110B: Elements of Partial Differential Equations and Integral Equations
Winter quarter, 2017

Homework Assignment 1

Due Friday, January 13

1. Let $u = u(x, y, z)$ be a twice differentiable function. Prove that $\nabla \cdot \nabla u = \Delta u$.

2. Let α be a real number, and define the change of coordinates $x' = x\cos\alpha - y\sin\alpha$ and $y' = x\sin\alpha + y\cos\alpha$. Let $u = u(x, y)$ be a twice differentiable function. Let $v(x', y') = u(x, y)$ under the defined change of coordinates. Prove that $\Delta u(x, y) = \Delta v(x', y')$.

3. Let $u = u(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$. Assume u is twice differentiable in x, y, and z. Prove that
$$\Delta u = u_{rr} + \frac{2}{r}u_r = \frac{1}{r^2}\left(r^2 u_r\right)_r$$
in the region $r > 0$.

4. Let $u = u(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$. Solve the equation $\Delta u = 1$ in the region $1 < r < 2$ with the boundary conditions $u(1) = 1$ and $u(2) = 1$.

5. Let u be a harmonic function on the disk $D = \{r < 3\}$ and $u = 2\cos\theta + 3\sin\theta$ for $r = 3$.
   (1) Find the minimum value of u in $\overline{D}$.
   (2) Calculate the value of u at the origin.