Math 110B: Elements of Partial Differential Equations and Integral Equations
Winter quarter, 2017

Homework Assignment 2

Due Friday, January 20

1. For each integer $n \geq 1$, define $u_n, v_n : \mathbb{R}^2 \to \mathbb{R}$ in the polar coordinates by
$$u_n(r,\theta) = r^n \cos(n\theta) \qquad \text{and} \qquad v_n(r,\theta) = r^n \sin(n\theta).$$
Show that both u_n and v_n are harmonic functions in $\mathbb{R}^2$.

2. Consider the boundary-value problem
$$\begin{cases} \Delta u = 0 & \text{in } \{r > a\}, \\ u(a,\theta) = h(\theta) & \text{on } \{r = a\}, \\ u \text{ is bounded} & \text{as } r \to \infty, \end{cases}$$
where $h = h(\theta)$ is a continuous and 2π-periodic function. The solution formula for this problem is given in Eq. (9) on page 175 of the textbook. (See more details in Example 3 on pages 174 and 175 of the textbook.)
   (1) What is the difference between this formula and Poisson's formula, Eq. (13) on page 168 of the textbook?
   (2) Why the right-hand side of Eq. (9) as a function of (r,θ) is bounded as $r \to \infty$?

3. Let D be a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a smooth boundary ∂D. Let $f : \overline{D} \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ be two continuous functions. Assume $u : \overline{D} \to \mathbb{R}$ has all the continuous, second-order partial derivatives, and solves the boundary-value problem
$$\begin{cases} \Delta u = -f & \text{in } D, \\ \partial_n u = g & \text{on } \partial D, \end{cases}$$
where ∂_n is the partial derivative along the exterior normal n along the boundary ∂D. Prove that
$$\int_D f\,dx + \int_{\partial D} g\,dS = 0.$$
(Hint: See Eq. (3) and Eq. (4) on page 180 of the textbook.)

4. Let D be a bounded domain in R^d ($d = 2$ or 3) with a smooth boundary ∂D and f a given function on D. Prove the uniqueness of solution to the boundary-value problem of $\Delta u = f$ in D with
   (1) the Dirichlet boundary condition $u = g$ on ∂D, where g is a continuous function on ∂D, or
   (2) with the Robin boundary condition $\partial_n u + au = b$ on ∂D, where a and b are two continuous functions on ∂D and a is strictly positive.

   What about the Neumann boundary condition $\partial_n u = h$ for some continuous function h on ∂D?