1. For each integer $n \geq 1$, define $u_n, v_n : \mathbb{R}^2 \to \mathbb{R}$ in the polar coordinates by

\[
u_n(r, \theta) = r^n \cos(n\theta) \quad \text{and} \quad v_n(r, \theta) = r^n \sin(n\theta).
\]

Show that both $u_n$ and $v_n$ are harmonic functions in $\mathbb{R}^2$.

2. Consider the boundary-value problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \{r > a\}, \\
u(a, \theta) = h(\theta) & \text{on } \{r = a\}, \\
u & \text{is bounded as } r \to \infty,
\end{cases}
\]

where $h = h(\theta)$ is a continuous and $2\pi$-periodic function. The solution formula for this problem is given in Eq. (9) on page 175 of the textbook. (See more details in Example 3 on pages 174 and 175 of the textbook.)

(1) What is the difference between this formula and Poisson’s formula, Eq. (13) on page 168 of the textbook?

(2) Why the right-hand side of Eq. (9) as a function of $(r, \theta)$ is bounded as $r \to \infty$?

3. Let $D$ be a bounded domain in $\mathbb{R}^d$ ($d = 2$ or $3$) with a smooth boundary $\partial D$. Let $f : \overline{D} \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ be two continuous functions. Assume $u : \overline{D} \to \mathbb{R}$ has all the continuous, second-order partial derivatives, and solves the boundary-value problem

\[
\begin{cases}
\Delta u = -f & \text{in } D, \\
\partial_n u = g & \text{on } \partial D,
\end{cases}
\]

where $\partial_n$ is the partial derivative along the exterior normal $n$ along the boundary $\partial D$. Prove that

\[
\int_D f \, dx + \int_{\partial D} g \, dS = 0.
\]

(Hint: See Eq. (3) and Eq. (4) on page 180 of the textbook.)

4. Let $D$ be a bounded domain in $R^d$ ($d = 2$ or $3$) with a smooth boundary $\partial D$ and $f$ a given function on $D$. Prove the uniqueness of solution to the boundary-value problem of $\Delta u = f$ in $D$ with

(1) the Dirichlet boundary condition $u = g$ on $\partial D$, where $g$ is a continuous function on $\partial D$,

(2) with the Robin boundary condition $\partial_n u + au = b$ on $\partial D$, where $a$ and $b$ are two continuous functions on $\partial D$ and $a$ is strictly positive.

What about the Neumann boundary condition $\partial_n u = h$ for some continuous function $h$ on $\partial D$?