Math 130B: Ordinary Differential Equations II, Spring 2015

Practice for Final Exam

1. What is a gradient system? Prove that a gradient system does not have a closed orbit.

2. Consider $\dot{x} = y^2 + y\cos x$, $\dot{y} = 2xy + \sin x$. Determine if this is a gradient system. If it is, then find the potential $V = V(x, y)$ so that $\dot{x} = -\partial_x V$ and $\dot{y} = -\partial_y V$.

3. Show that $V(x, y) = x^2 + y^2$ is a Liapunov function for the system $\dot{x} = y - x^3$ and $\dot{y} = -x - y^3$ at the fixed point $(0, 0)$.

4. Show by Dulac's Criterion that the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the first quadrant $x > 0$, $y > 0$.

5. Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

6. Consider the system $\dot{x} = x(1 - 4x^2 - y^2) - 0.5y(1 + x)$, $\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x)$.
   (a) Show that the origin is an unstable fixed point.
   (b) Consider $\dot{V}$ with $V(x, y) = (1 - 4x^2 - y^2)^2$ to show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

7. Consider the system $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$.
   (a) Write the system in the polar coordinates. (You can use $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (\dot{y}x - \dot{x}y)/r^2$.)
   (b) Use the trapping region method to show that this system has a closed orbit in the region defined by $r_1 < r < r_2$ for some positive numbers r_1 and r_2 with $0 < r_1 < r_2$.

8. Consder the two-timing expansion $x(t, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2)$, where $\tau = t$ and $T = \varepsilon t$. Plug this expansion into the equation $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$ to get the equations for x_0 and x_1.

9. Consider $\dot{x} = y - 2x$, $\dot{y} = \mu + x^2 - y$.
   (a) Sketch the nullclines.
   (b) Find and classify the bifurcations that occur as μ varies.
   (c) Sketch the phase portrait as a function of μ.

10. Consider the logistic equation $\dot{N} = rN(1 - N/K)$, where $K = K(t)$ is smooth, positive, and T-periodic. Use the Poincaré map argument to show that this system has at least one limit cycle of period T, contained in the strip $K_{\min} \le N \le K_{\max}$.

11. Consider the system in polar coordinates $\dot{r} = \mu r - r^3$ and $\dot{\theta} = 1$, where μ is a parameter. In the cartesian coordinates, this system is $\dot{x} = (\mu - x^2 - y^2)x - y$ and $\dot{y} = (\mu - x^2 - y^2)y + x$.
    (a) Suppose $\mu < 0$. Show that the origin is a stable spiral and explain why all trajectories approach the origin.
    (b) Suppose $\mu > 0$. Show that the origin is an unstable spiral and explain why the circle $r = \sqrt{\mu}$ is a stable limit cycle.

    (This system has a supercritical Hopf bifurcation at $\mu_c = 0$.)

12. Consider the vector field given in the polar coordinates by $\dot{r} = r - r^2$, $\dot{\theta} = 1$.
    (a) Let S be the positive x-axis and compute the Poincaré map from S to itself.
    (b) Show that the system has a unique periodic orbit and classify its stability.

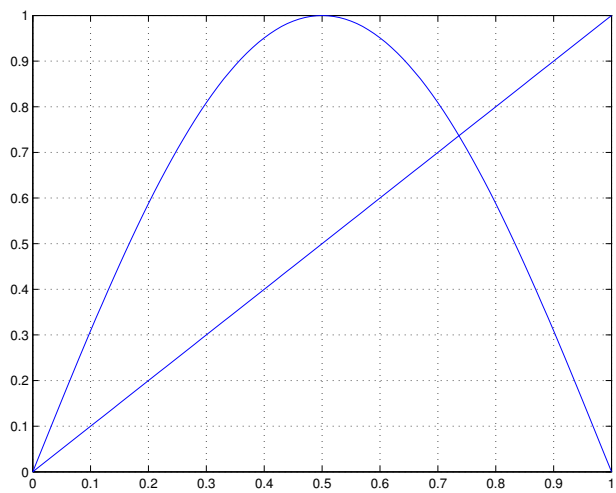
13. Prove for the Lorenz system that volume in the phase space contracts.

14. Prove that the z-axis is an invariant line for the Lorenz system, i.e., a trajectory starts on the z-axis stays on it forever.

15. Draw the cobweb for $x_{n+1} = \sin x_n$ for $n = 0, 1, 2, 3$ with $x_0 = \pi/2$.

16. Draw the cobweb for $x_{n+1} = \cos x_n$ for $n = 0, 1, 2, 3$ with $x_0 = -\pi/2$.

17. The figure below shows the graphs of $y = x$ and $y = \sin(\pi x)$ Set $x_0 = 0.4$ and $x_{n+1} = \sin(\pi x_n)$ $(n = 0, 1, 2, \dots)$. Construct the cobweb to find x_1, x_2, x_3, x_4. These numbers should be marked on the x-axis.



18. Show that $x_{n+1} = 1 + (1/2)\sin x_n$ has a unique fixed point. Is it stable?

19. Let $0 < a < e^{-1}$. Show that the map $x_{n+1} = ae^{x_n}$ has exactly two fixed points and classify their linear stabilities.

20. Show that the logistic map $x_{n+1} = rx_n(1 - x_n)$ has a 2-cycle if $r > 3$.

21. Consider a smooth map $x_{n+1} = f(x_n)$. Let $k \geq 2$ be any integer. Prove the following:
    (a) If x^* is a fixed point of f, then it is also a fixed point of f^k.
    (b) The derivative $(d/dx)(f^k)(x_0) = \Pi_{i=0}^{k-1} f'(x_i)$, where $x_{i+1} = f(x_i)$.

22. The map $x_{n+1} = 1 - rx_n^2$ has a superstable 3-cycle for certain value of r. Find a cubic equation for this r.

23. Calculate the Liapunov exponent for the linear map $x_{n+1} = rx_n$.

24. Use the fact that the set of all real numbers is uncountable and that the set of all rational numbers is countable to prove that the set of irrational numbers is uncountable.

25. Calculate the dimension for the middle-thirds Cantor set as a self-similar fractal.

26. Calculate the box dimension of the von Koch curve.