Math 130B: Ordinary Differential Equations and Dynamical Systems (II), Spring 2019

Review for Final Exam

1. **Equilibrium points and their stabilities (Sections 8.1–8.4, and 9.2)**
   (a) Equilibrium points, and their stability, asymptotic stability, and instability.
   (b) Linearized systems. The Linearization Theorem for hyperbolic equilibrium points.
   (c) Liapunov functions and the Liapunov Stability Theorem.

2. **Bifurcation (Section 8.5)**
   (a) Saddle-node (e.g., $\dot{x} = r - x^2$, $\dot{y} = -y$), transcritical (e.g., $\dot{x} = rx - x^2$, $\dot{y} = -y$), and pitchfork bifurcations (e.g., supercritical: $\dot{x} = rx - x^3$, $\dot{y} = -y$, subcritical: $\dot{x} = rx + x^3$, $\dot{y} = -y$): model equations, bifurcation diagrams, and phase portraits.
   (b) Hopf bifurcations: creation or annihilation of closed orbits.

3. **Gradient systems and (planar Hamiltonian systems (Sections 9.3 and 9.4)**
   (a) Gradient systems. A necessary condition. Equilibrium points and their stabilities.
   (b) Why $(d/dt)V(x(t)) \leq 0$? Why a gradient system has no non-constant period solutions?
   (c) Why trajectories of solutions to a gradient system $\dot{x} = -\nabla V(x)$ are orthogonal to the level surfaces of the potential function V?
   (d) Show that the Hamiltonian on any trajectory is a constant: $(d/dt)H(x(t), y(t)) = 0$.
   (e) Why a solution to a Hamiltonian system will stay in a level curve of the Hamiltonian?

4. **Closed orbits and limit sets (Sections 10.1–10.3, 10.5, 12.2, and 12.3)**
   (a) The α-limit set and ω-limit set of a point x (or a trajectory).
   (b) The Dulac's criterion for excluding closed orbits.
   (c) Construction of Poincaré maps. Use the Poincaré map to study the stability of a closed orbit.
   (d) The Poincaré–Bendixson Theorem. The trapping region method.
   (e) The Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, equivalently: $\dot{x} = y - F(x)$ and $\dot{y} = -g(x)$ with $F'(x) = f(x)$. The van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$.

5. **The Lorenz System (Sections 14.1–14.3)**
   (a) Why the change of variables $(x, y, z) \to (-x, -y, z)$ does not change the Lorenz equations?
   (b) Why the Lorenz system is dissipative (i.e., volume contraction)?
   (c) Show that the z-axis is invariant for the Lorenz system.
   (d) Show that any trajectory of solution to the Lorenz equations will eventually enter into and stay inside a sphere centered at $(0, 0, r + \sigma)$.
   (e) Fixed points $(0, 0, 0)$ for $r < 1$, $(0, 0, 0)$ and Q_+ and Q_- for $r > 1$, and their stabilities.

6. **One-dimensional maps (Sections 15.1–15.3)**
   (a) Fixed points and their stabilities. Periodic orbits and their stabilities. Web diagrams.
   (b) Bifurcations.
   (c) The discrete logistic model: $x_{n+1} = \lambda x_n(1 - x_n)$.

Additional Practice Problems

1. For each of the following systems, find all the fixed points and the corresponding linearized systems, and use the linearization theorem to classify these fixed points and determine their stabilities:
   (a) $\dot{x} = x - y$, $\dot{y} = x^2 - 4$;

(b) $\dot{x} = xy - 1$, $\dot{y} = x - y^3$;

(c) $\dot{x} = y^2 - 3x + 2$, $\dot{y} = x^2 - y^2$.

2. Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.
   (a) Find all the fixed points and classify them.
   (b) Show that the line $x = y$ is invariant, i.e., any trajectory that starts on it stays on it.
   (c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all other trajectories. (Hint: Form a different equation for $x - y$.)
3. Show that $V(x, y) = x^2 + y^2$ is a Liapunov function for the system $\dot{x} = y - x^3$ and $\dot{y} = -x - y^3$ at the equilibrium point $(0, 0)$. Is this equilibrium point stable? asymptotically stable?
4. Find a strict Liapunov function for the equilibrium point $(0, 0)$ of $\dot{x} = -2x - y^2$, $\dot{y} = -y - x^2$.
5. Show that the first-order system $\dot{x} = r - x - e^{-x}$, $\dot{y} = -y$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point.
6. Consider the equation $\dot{x} = 1 + rx + x^2$, where $r \in \mathbb{R}$ is a parameter.
   (a) Find the critical value $r = r_c$ at which a saddle-node bifurcation occurs.
   (b) Sketch the birfaction diagram of the fixed point x vs. r.
7. Consider the equation $\dot{x} = x - rx(1 - x)$, where $r \in \mathbb{R}$ is a parameter.
   (a) Find the critical value $r = r_c$ at which a transcritical bifurcation occurs.
   (b) Sketch the bifurcation diagram of the fixed point x vs. r.
8. Consider the equation $\dot{x} = x + rx/(1 + x^2)$, where $r \in \mathbb{R}$ is a parameter.
   (a) Find the critical value $r = r_c$ at which a pitchfork bifurcation occurs.
   (b) Sketch the bifurcation diagram of the fixed point x vs. r.
9. Consider $\dot{x} = y - 2x$, $\dot{y} = \mu + x^2 - y$.
   (a) Sketch the nullclines.
   (b) Find and classify the bifurcations that occur as μ varies.
   (c) Sketch the phase portrait as a function of μ.
10. The system $\dot{x} = (\mu - x^2 - y^2)x - y$, $\dot{y} = (\mu - x^2 - y^2)y + x$, where μ is a parameter, can be reformulated in the polar coordinates as $\dot{r} = \mu r - r^3$ and $\dot{\theta} = 1$. Show this system undergoes a Hopf bifurcation at $\mu = 0$ by showing the following:
    (a) If $\mu < 0$ then the origin is a stable spiral, and all trajectories approach the origin.
    (b) If $\mu > 0$ then the origin is an unstable spiral, and the circle $r = \sqrt{\mu}$ is a stable periodic solution.
11. Let $V(x, y) = (x^2 - 1)^2 + y^2$. Find all the equilibrium points and determine their stabilities for the gradient system $\dot{x} = -\partial_x V(x, y)$, $\dot{y} = -\partial_y V(x, y)$. Draw a few level curves $V(x, y) = c$ and draw some trajectories.
12. Identify all the points that lie in either an ω-limit set or an α-limit set of a solution trajectory of the system $\dot{r} = r^3 - 3r^2 + 2r$, $\dot{\theta} = 1$.
13. Show by Dulac's Criterion that the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the first quadrant $x > 0$, $y > 0$.
14. Use Dulac's criterion to show that the system $\dot{x} = y$, $\dot{y} = -x - y + x^2 + y^2$ has no closed orbits. (Hint: try $g(x, y) = e^{-2x}$.)
15. Consider the system $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$.
    (a) Write the system in the polar coordinates. (You can use $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (\dot{y}x - \dot{x}y)/r^2$.)
    (b) Use the trapping region method to show that this system has a closed orbit in the region defined by $r_1 < r < r_2$ for some positive numbers r_1 and r_2 with $0 < r_1 < r_2$.
16. Consider the two-dimensional system $\dot{X} = AX - \|X\|^2 X$, where A is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\beta$ ($\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$). Prove that there exists at least one limit cycle if $\alpha > 0$ and that there are none if $\alpha < 0$.
17. Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.
18. Consider the system $\dot{x} = x(1 - 4x^2 - y^2) - 0.5y(1 + x)$, $\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x)$.
    (a) Show that the origin is an unstable fixed point.

(b) Consider $\dot{V}$ with $V(x,y) = (1-4x^2-y^2)^2$ to show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

19. Consider the three-dimensional system in the cylindrical coordinates $\dot{r} = r(1-r)$, $\dot{\theta} = 1$, $\dot{z} = -z$. Compute the Poincaré map $P : S \to S$ along the closed orbit $r = 1$, $z = 0$, where $S = \{(\theta, r, z) : \theta = 0, r > 0, z \in \mathbb{R}\}$, and show that this closed orbit is asymptotically stable.
20. Consider the three-dimensional system in the cylindrical coordinates $\dot{r} = r(1-r)$, $\dot{\theta} = 1$, $\dot{z} = z$. Compute again the Poincaré map $P : S \to S$ along the closed orbit $r = 1$, $z = 0$. where $S = \{(\theta, r, z) : \theta = 0, r > 0, z \in \mathbb{R}\}$, What can you now say about the behavior of solutions near the closed orbit? Sketch the phase portrait for this system.
21. The system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ has a closed orbit $r = 1$. Compute the Poincaré map defined on the positive x-axis for this orbit and determine the stability of this orbit.
22. Consider the system in polar coordinates $\dot{r} = \mu r - r^3$ and $\dot{\theta} = 1$, where μ is a parameter. In the cartesian coordinates, this system is $\dot{x} = (\mu - x^2 - y^2)x - y$ and $\dot{y} = (\mu - x^2 - y^2)y + x$.
    (a) Suppose $\mu < 0$. Show that the origin is a stable spiral and explain why all trajectories approach the origin.
    (b) Suppose $\mu > 0$. Show that the origin is an unstable spiral and explain why the circle $r = \sqrt{\mu}$ is a stable limit cycle.

    (This system has a supercritical Hopf bifurcation at $\mu_c = 0$.)
23. Consider the vector field given in the polar coordinates by $\dot{r} = r - r^2$, $\dot{\theta} = 1$.
    (a) Let S be the positive x-axis and compute the Poincaré map from S to itself.
    (b) Show that the system has a unique periodic orbit and classify its stability.
24. Prove the elementary properties for the Lorenz system: volume contraction; existence of a trapping sphere; and the z-axis is invariant.
25. Draw the cobweb for $x_{n+1} = \sin x_n$ for $n = 0, 1, 2, 3$ with $x_0 = \pi/2$.
26. Find all the fixed points for the one-dimensional map $x_{n+1} = f(x_n)$ and determine their stabilities, where the function $f(x)$ is given by: (a) $f(x) = x^2$; (b) $f(x) = 3x - x^3$.
27. Show that $x_{n+1} = 1 + (1/2)\sin x_n$ has a unique fixed point. Is it stable?
28. Let $0 < a < e^{-1}$. Show that the map $x_{n+1} = ae^{x_n}$ has exactly two fixed points and classify their linear stabilities.
29. Show that the logistic map $x_{n+1} = rx_n(1-x_n)$ has a 2-cycle if $r > 3$.
30. Show that the iteration $x_{n+1} = \cos x_n$ has a unique fixed point. Predict the stability of this fixed point by drawing the web diagram. Can you prove your statement?
31. Find possible periodic points with period $n = 1$ or 2 for each of the following maps and classify them as attracting, repelling, or neither: (a) $f(x) = x - x^2$; (b) $f(x) = 2(x - x^2)$.
32. Discuss the bifurcations that occur in the following families of maps at the indicated parameter value: (a) $S_\lambda(x) = \lambda \sin x$ at $\lambda = 1$; (b) $E_\lambda(x) = \lambda e^x$ at $\lambda = 1/e$.
33. Assume that x_0 lies on a cycle of period n for the map defined by $f(x)$. Let $x_i = f(x_{i-1})$ $(i = 1, \ldots, n)$. Show that $(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$, and conclude that

$$(f^n)'(x_0) = (f^n)'(x_j), \qquad j = 1, \ldots, n-1.$$