Hints/Soln's to Practice Problems

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1 (a) $x=y$. $x=\pm 2$. $\boxed{(2,2),\ (-2,-2)}$

$$A=\begin{bmatrix}1 & -1\\ 2x & 0\end{bmatrix}_{(x,y)=(2,2)} = \begin{bmatrix}1 & -1\\ 4 & 0\end{bmatrix}$$

$(\lambda-1)\lambda+4=0$ $\lambda^2-\lambda+4=0$, $\lambda_{1,2}=\frac{1\pm\sqrt{1-16}}{2}$

$\boxed{(2,2) \text{ is a spiral, unstable}}$

$$A=\begin{bmatrix}1 & -1\\ 2x & 0\end{bmatrix}_{(-2,-2)} = \begin{bmatrix}1 & -1\\ -4 & 0\end{bmatrix}$$

$(\lambda-1)\lambda-4=0$ $\lambda_{1,2}=\frac{1\pm\sqrt{1+16}}{2}$

$\lambda^2-\lambda-4=0$

$\boxed{(-2,-2) \text{ is a saddle pt. unstable}}$

(b) $xy=1 \Rightarrow y=\frac{1}{x}$. $x=y^3 \Rightarrow x=\frac{1}{x^3}$. $x^4-1=0$

$x=\pm 1$. $y=\pm 1$. $\boxed{(1,1),\ (-1,-1)}$

$$A=\begin{bmatrix}y & x\\ 1 & -3y^2\end{bmatrix}_{(1,1)} = \begin{bmatrix}1 & 1\\ 1 & -3\end{bmatrix}$$ $(\lambda-1)(\lambda+3)-1=0$

$\lambda^2+2\lambda-4=0$

$\lambda_{1,2}=\frac{-2\pm\sqrt{4+16}}{2}=-1\pm\sqrt{5}$.

$\boxed{(1,1) \text{ is a saddle pt, unstable}}$

$$A(-1,-1)=\begin{bmatrix}-1 & -1\\ 1 & -3\end{bmatrix}$$ $(\lambda+1)(\lambda+3)+1=0$

$\lambda^2+4\lambda+4=0$

$(\lambda+2)^2=0$. $\lambda_{1,2}=-2$.

$\boxed{(-1,-1): \text{ sink. (degenerate) } \cancel{\text{un}}\text{stable}}$

(c) $y=\pm x$. $x^2-3x+2=0$ $x=1,\ x=2$

$(1,1),\ (1,-1),\ (2,2),\ (2,-2)$.

$$A = \begin{bmatrix} -3 & 2y \\ 2x & -2y \end{bmatrix}$$

$(1,1)$: $A = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}$ $D = \det A = 6-4 = 2 > 0$
$T = Tr(A) = -5 < 0$

$T^2 - 4D = 25 - 8 > 0$ $\boxed{(1,1): \text{sink}}$ stable

$(1,-1)$ $A = \begin{bmatrix} -3 & -2 \\ 2 & -2 \end{bmatrix}$ $D = -2 < 0$ $T = -1 < 0$. $\boxed{(1,-1): \text{saddle unstable}}$

$(2,2)$: $A = \begin{bmatrix} -3 & 4 \\ 4 & -4 \end{bmatrix}$ $D = -4$, $T = -7$

$T^2 - 4D = 49 + 16 > 0$. $\boxed{(2,2): \text{saddle unstable}}$

$(2,-2)$ $A = \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix}$ $D = 4 > 0$
$T = 1 > 0$. $T^2 - 4D < 0$.

$\boxed{(2,-2): \text{spiral out, unstable}}$

2 (a) $x = \frac{1}{4}y^3$ $y^3 - y - \frac{3}{4}y^3 = 0$. $\frac{1}{4}y^3 - y = 0$. $y(y^2-4) = 0$

$y = 0, 2, -2$. $x = 0, 2, -2$. $\boxed{(0,0), (2,2), (-2,-2)}$

$$A = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2-1 \end{bmatrix}$$

$(0,0)$ $A = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}$ $\lambda_1 = -4, \lambda_2 = -1$

$\boxed{(0,0): \text{sink, stable}}$

$(2,2)$ and $(-2,-2)$ $A = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}$ $(\lambda+4)(\lambda-11) + 36 = 0$
$\lambda^2 - 7\lambda - 8 = 0$
$(\lambda-8)(\lambda+1) = 0$. $\lambda_1 = 8$, $\lambda_2 = -1$

$\boxed{(2,2) \text{ and } (-2,-2): \text{saddle, unstable}}$

(b) $\dot{x} - \dot{y} = y^3 - 4x - y^3 + y + 3x = y - x = -(x-y)$.

So $x(t) - y(t) = (x(0) - y(0))e^{-t}$.

If $x(0) = y(0)$, then $x(t) = y(t)$ $\forall t$.

(c) From (b), $|x(t)-y(t)| = |x(0)-y(0)|e^{-t} \to 0$ as $t \to +\infty$.

3. $V(x,y) > 0$ if $(x,y) \neq (0,0)$. $V(0,0)=0$.

$\dot{V} = \partial_x V\, x' + \partial_y V\, y' = 2x(y-x^3) + 2y(-x-y^3)$

$= -2x^4 - 2y^4 < 0$ if $(x,y) \neq (0,0)$.

So, V is a strict Liapunov function at $(0,0)$.

$(0,0)$ is a stable and asymptotically stable equilibrium point.

4. Let $V(x,y) = x^2+y^2$. $V>0$ except at $(0,0)$

$\dot{V} = \partial_x V\, x' + \partial_y V\, y' = 2x(-2x-y^2) + 2y(-y-x^2)$

$= -4x^2 - 2xy^2 - 2y^2 - 2yx^2$

$= -3x^2 - y^2 - (x^2+y^2+2xy^2+2yx^2)$

$= -3x^2 - y^2 - \cancel{2}(x^2+y^2)\left(1+\frac{2(xy^2+yx^2)}{x^2+y^2}\right)$

Since $\left|\frac{2(xy^2+yx^2)}{x^2+y^2}\right| \le \left|\frac{2xy}{x^2+y^2}\right|(|y|+|x|) \to 0$

$\Rightarrow \exists\, \delta>0$. $0<x^2+y^2<\delta^2 \Rightarrow \cancel{1+}\left|\frac{2(xy^2+yx^2)}{x^2+y^2}\right| < \frac{1}{2}$.

Hence, $\forall (x,y) \in D_\delta$: $x^2+y^2<\delta^2$, $(x,y)\neq(0,0)$

$\Rightarrow \dot{V} \le -3x^2 - y^2 - (x^2+y^2)\left(1-\frac{1}{2}\right) < 0$.

Hence V is a strict Liapunove function at $(0,0)$

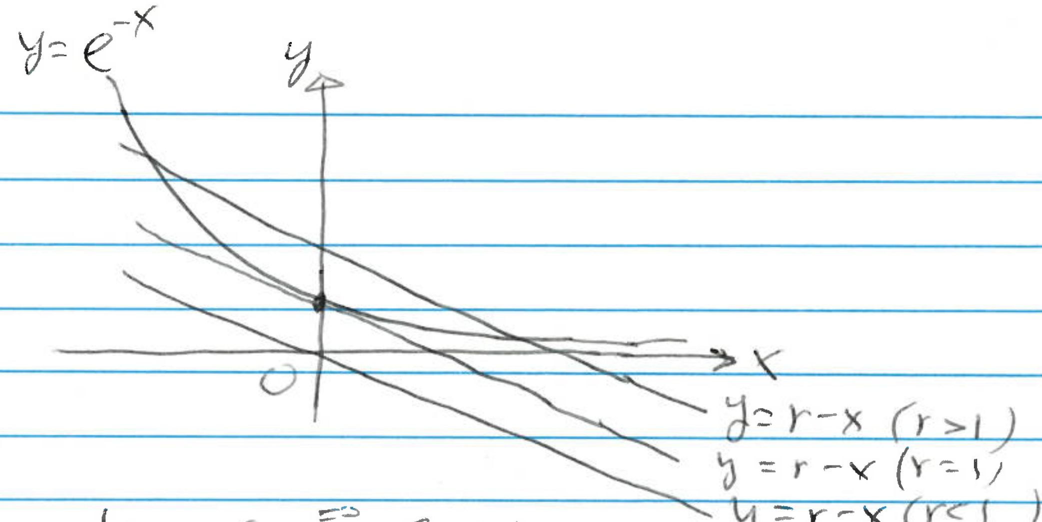
5. $r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2!} + \cdots\right) = r - 1 - \frac{x^2}{2} + \cdots$

So $r_c = 1$.

$r<1$. ~~Fct~~. $g(x) = x + e^{-x}$ is minimized at $x=0$ with minimum value $g(0) = 1 > r$. ($g'(x) = 1-e^{-x}$, $g'(0)=0$. $g''(x) = e^{-x} > 0$.)

$r=1$. $x=0$ is the unique equilibrium point. Since the minimum of above g is unique.

$r>1$. Exactly two roots of $f(x) := r - x - e^{-x} = 0$.



Argue by. $f(x)=0$ on $[0,\infty)$; $f(0)>0$. $f(+\infty)=-\infty$. $f'<0$.

$f(x)=0$ on $[0,-\infty)$; $f(0)>0$ $f(-\infty)=-\infty$. $f'>0$.

each has exactly one solution

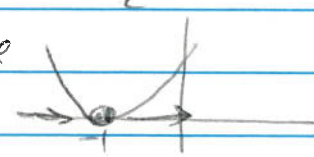
6. (a) $x^2+rx+1=0$ $\quad x_{1,2}=\frac{-r\pm\sqrt{r^2-4}}{2}$

$r_c=2$ ~~and~~ or -2.

(b) $r_c=2$. $r<r_c$ no equilibrium pts

$r=r_c=2$. $x_{1,2}=\frac{-2\pm\sqrt{0}}{2}=-1$.

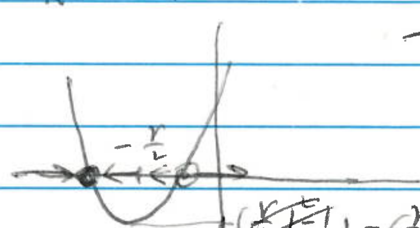
one equil. pt. semi-stable



$r>r_c=2$

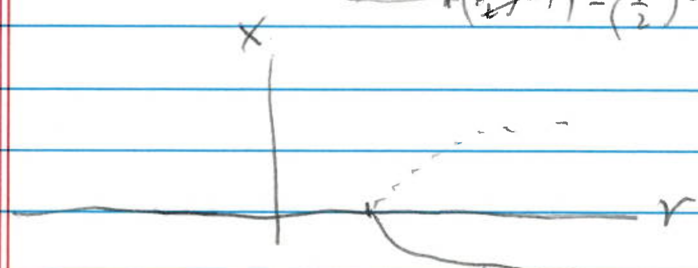
two equil pts $x_{1,2}=\frac{-r\pm\sqrt{r^2-4}}{2}$

$x'=x^2+rx+(\frac{r}{2})^2+1-(\frac{r}{2})^2=(x+\frac{r}{2})^2-\left((\frac{r}{2})^2-1\right)$



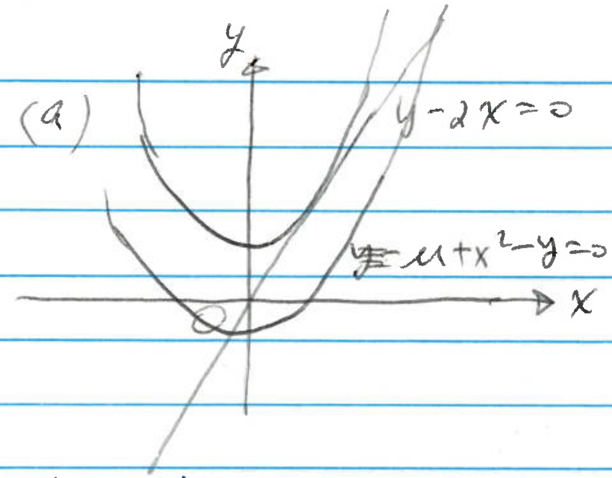
$\frac{-r+\sqrt{r^2-4}}{2}$ unstable

$\frac{-r-\sqrt{r^2-4}}{2}$ stable.



For $r_c=-2$: similar

9. (a)



(b) $y=2x \Rightarrow \mu+x^2-y=0$. $\mu+x^2-2x=0$

$x^2-2x+\mu=0$ $\quad x_{1,2}=\frac{1}{2}\left(2\pm\sqrt{4-4\mu}\right)$

$\mu=1$:

$\mu>1$: no equil. pts.

$\mu=1$. one equil pt. $x=1 \Rightarrow y=2$. $(1,2)$.

$\mu<1$. two equil. pts.

saddle-node

10. (a) $\mu<0$. $\quad \dot{r}=\mu r-r^3$.

$(0,0)$. Equil. pt. linearized system $\dot{x}=\mu x-y$

$\dot{y}=\mu y+x$

$A=\begin{bmatrix}\mu & -1\\ 1 & \mu\end{bmatrix}$ $\quad (\lambda-\mu)^2+1=0$

$\lambda_{1,2}=\mu\pm i$ $\quad \mu<0$.

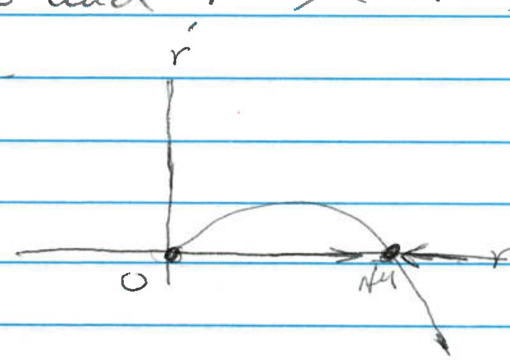
So spiral in. stable

(b) $\mu>0$. Linearized system: same. $\lambda_{1,2}=\mu\pm i$

$\mu>0 \Rightarrow$ spiral out

$\dot{r}=\mu r-r^3=0 \Rightarrow r=0$ and $r^2=\mu \Rightarrow r=\sqrt{\mu}$

is a periodic solution



11. $\partial_x V = 2(x^2-1)\,2x \qquad \partial_y V = 2y.$

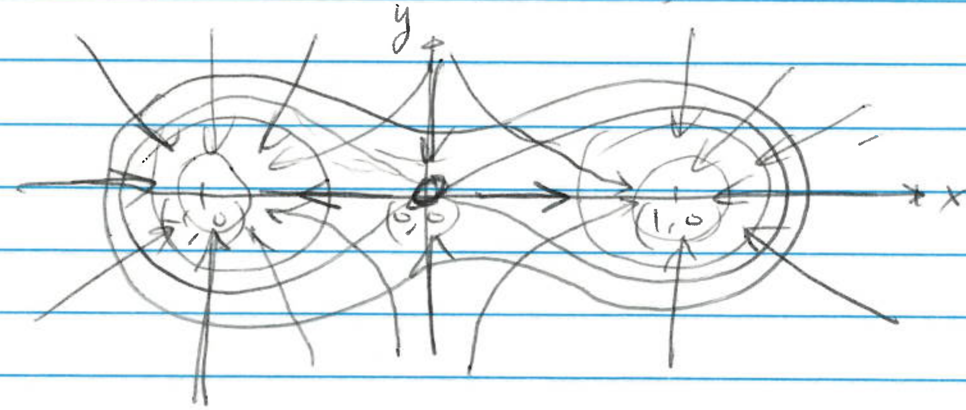
$\partial_x V = 0,\ \partial_y V = 0 \Longrightarrow y=0,\ x=0$ or $x=\pm 1$

Equil. pts. $(0,0),\ (1,0),\ (-1,0)$

$$A = \begin{bmatrix} -\partial_{xx}V & -\partial_{xy}V \\ -\partial_{xy}V & -\partial_{yy}V \end{bmatrix} = \begin{bmatrix} -4(3x^2-1) & 0 \\ 0 & -2 \end{bmatrix}$$

$(0,0)$: $A = \begin{bmatrix} +4 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad \lambda_1 = +4,\ \lambda_2 = -2.$ saddle, unstable

$(1,0)$ and $(-1,0)$ $\quad A = \begin{bmatrix} -8 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad$ sink, stable



12. $r = 0,\ 1,\ 2.$

13. $g(x,y) = \frac{1}{xy}.$

$$\nabla\cdot(g\dot{X}) = \nabla\cdot\left(\frac{1}{xy}\begin{bmatrix} x(2-x-y) \\ y(4x-x^2-3)\end{bmatrix}\right)$$

$$= \frac{1}{y}(-1) + \frac{1}{x}0 = -\frac{1}{y} < 0 \quad (x, y > 0)$$

Dulac's Criterion

1. R — simply connected region
2. $\nabla\cdot(g\dot{X})$ same sign on R

$\Longrightarrow$ no closed orbit in R

14. Similar.