Math 130B: ODE and Dynamical Systems, Spring 2019

Homework Assignment 6

Due Monday, May 20, 2019

1. Use Dulac's criterion to show that the system $\dot{x} = y$, $\dot{y} = -x - y + x^2 + y^2$ has no closed orbits. (Hint: try $g(x, y) = e^{-2x}$.)

2. Consider the system $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$.
   (1) Write the system in the polar coordinates. (You can use $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (\dot{y}x - \dot{x}y)/r^2$.)
   (2) Use the trapping region method to show that this system has a closed orbit in the region defined by $r_1 < r < r_2$ for some positive numbers r_1 and r_2 with $0 < r_1 < r_2$.

3. Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

4. Consider the two-dimensional system $\dot{X} = AX - \|X\|^2 X$, where A is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\beta$ ($\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$). Prove that there exists at least one limit cycle if $\alpha > 0$ and that there are none if $\alpha < 0$.

5. Use the Liénard theorem to show that the van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has a unique stable limit cycle for any parameter $\mu > 0$.