Math 130B: ODE and Dynamical Systems, Spring 2019

Homework Assignment 7

Due Wednesday, May 29, 2019

1. Show that the system $\dot{x} = -\nu x + zy$, $\dot{y} = -\nu y + (z - a)x$, $\dot{z} = 1 - xy$, where $a, \nu > 0$ are parameters, is dissipative (i.e., the volume is contractive).

2. (A spherical trapping region for the Lorenz system.) Show that all trajectories of the Lorenz system eventually enter and remain inside a large sphere S of the form $x^2 + y^2 + (z - r - \sigma)^2 = C$ for C sufficiently large. (Hint: Show that $x^2 + y^2 + (z - r - \sigma)^2$ decreases along trajectories for all (x, y, z) outside a certain fixed ellipsoid. Then pick C large enough so that the sphere S encloses this ellipsoid.)

3. Consider the Lorenz equations.

   (1) Show that the characteristic equation for the eigenvalues of the Jacobian matrix at $C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $C^- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ is
   $$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

   (2) By seeking solutions of the form $\lambda = i\omega$, where ω is real, show that there is a pair of pure imaginary eigenvalues when
   $$r = r_H = \sigma \left(\frac{\sigma + b + 3}{\sigma - b - 1} \right).$$

   (3) Find the third eigenvalue.

4. Show that the z-axis is an invariant line for the Lorenz equations. (In other words, a trajectory that starts on the z-axis stays on it forever.)