MATH 132B, Spring Quarter, 2005
Instructor: Bo Li

Homework Assignment 2 (Due: Friday, May 13)

1. Let $x > 0$, $y > 0$, and $t > 0$. Calculate
$$I(x, y, t) = \int_0^\infty \sin(\alpha x) \sin(\alpha t) \frac{e^{-\alpha y}}{\alpha} d\alpha.$$
(Hint: Differentiate with resprect to x. See Page 18 of the textbook for the anwser.)
2. Show that for any real number λ, the Volterra integral equation of the second kind
$$u(x) = \lambda \int_0^x u(t)\, dt \qquad (0 \le x \le 1)$$
does not have any non-trivial solution $u = u(x)$ $(0 \le x \le 1)$ that has a continuous derivative on $[0, 1]$.
3. Let $u = u(x)$ be a real-valued function on $[0, 1]$. Suppose u has coninuous first and second derivatives on $[0, 1]$, and satisfis the integral equation
$$u(x) = \lambda \int_0^{1-x} u(t)\, dt \qquad (0 \le x \le 1),$$
where λ is a real nubmer.
   (1) Show that
$$u'' + \lambda^2 u = 0 \quad \text{for } 0 < x < 1,$$
$$u'(0) = 0 \quad \text{and} \quad u(1) = 0.$$
   (2) Show that there are infinitely many discrete values of λ such that the integral equation has non-trivial solutions with the assumed continuity property.
4. Solve the Fredholm integral equation of the second kind
$$u(x) - \lambda \int_0^\pi \sin(x + t) u(t)\, dt = f(x) \qquad (0 \le x \le 1)$$
with (1) $f(x) = 1$ and (2) $f(x) = \sin x + \cos x$. (Hint: See Example 3.1 on Page 59 and Example 3.2 on Page 62 of the textbook.)
5. Let $n \ge 1$ be an integer. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, define
$$\|\xi\|_1 = \sum_{j=1}^n |\xi_j| \qquad \text{and} \qquad \|\xi\|_\infty = \max_{1 \le j \le n} |\xi_j|.$$
   (1) Prove that both $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$.
   (2) Let $n = 2$. Draw on the $\xi_1\xi_2$-plane the sets
$$C_1 = \{\xi \in \mathbb{R}^2 : \|\xi\|_1 = 1\} \qquad \text{and} \qquad C_\infty = \{\xi \in \mathbb{R}^2 : \|\xi\|_\infty = 1\}.$$
6. For each integer $k \ge 1$, define $f_k : [0, 1] \to \mathbb{R}$ by $f_k(x) = -kx + 1$ if $0 \le x \le 1/k$ and $f_k(x) = 0$ if $1/k < x \le 1$. Show that the sequence $\{f_k\}_{k=1}^\infty$ converges to the zero function in $L^2[0, 1]$ but that it does not converge to any function in $C[0, 1]$.