MATH 132B, Spring Quarter, 2005
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Homework Assignment 2 (Due: Friday, June 3)

1. Let $k \in L^2([a,b] \times [a,b])$.
   (1) Let $u \in L^2[a,b]$ and

$$(Ku)(x) = \int_a^b k(x,t)u(t)\,dt.$$

   Prove that $Ku \in L^2[a,b]$.
   (2) Prove that $K : L^2[a,b] \to L^2[a,b]$, defined in Part (1), is a linear operator.
   (3) Prove that the linear operator $K : L^2[a,b] \to L^2[a,b]$ is continuous.
2. Let H be a Hilbert space and $\mathcal{B}(H)$ the Banach space of all the bounded linear operators on H. Suppose $T \in \mathcal{B}(H)$ and $\|T\| < 1$. Prove that following:
   (1) The series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{B}(H)$;
   (2) The operator $I - T \in \mathcal{B}(H)$ is invertible;
   (3) $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$.
3. Let H be a Hilbert space. Let $T : H \to H$ be a linear compact operator and $S : H \to H$ a bounded linear operator. Prove that both $ST : H \to H$ and $TS : H \to H$ are compact operators.