Math 170C: Numerical Analysis of Ordinary Differential Equations
Spring 2018

Review for Midterm Exam

Numerical Differentiation

1. The forward-difference ($h > 0$) and backward-difference ($h < 0$) formulas are given by

$$f'(x_0) = \frac{1}{h}\left[f(x_0 + h) - f(x_0)\right] - \frac{h}{2} f''(\xi).$$

   Exercise. Use the Taylor expansion to derive this formula.
   Exercise. If $f(0) = 1$ and $f(0.1) = 0.2$, what is an approximation of $f'(0)$? What is the maximal error (neglect the round-off error) of your approximation if $|f''(x)| \le 0.1$ for all x?

2. The central-difference formula (the three-point midpoint formula (4.5) on page 175) is

$$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] - \frac{h^2}{6} f'''(\xi).$$

   Exercise. Assume that f''' is continuous. Derive this formula using Taylor's expansion. You need to use $[f'''(\xi_1) + f'''(\xi_2)]/2 = f'''(\xi)$. Justify this.
   Exercise. If $f(0) = 1$, $f(-0.1) = 0.8$, and $f(0.1) = 0.2$, what is an approximation of $f'(0)$? If $|f'''(x)| \le 4$ for all x, what is a good error bound (neglect the round-off error)?
   Exercise. Study this formula in terms of round-off errors; cf. pages 178 and 179. Why the error now is controlled by $\varepsilon/h + h^2 M/6$?

3. The Lagrange interpolation leads to the $(n + 1)$-point numerical differentiation formula

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L_k'(x_j) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_j) \prod_{k=0, k\ne j}^{n} (x_j - x_k), \qquad j = 0, 1, \ldots, n,$$

   where $L_k(x) = (k = 0, \ldots, n)$ are the basic Lagrange polynomials associated with $x_0, x_1, \ldots, x_n$.
   Exercise. Derive the three-point ($n = 2$) endpoint formula (4.4) on page 175.
   Exercise. Suppose $f(0) = 0$, $f(0.1) = 0.2$, and $f(0.3) = 0.4$. Find an approximate of $f'(0)$.

4. The midpoint (also called the central-difference) formula for second-order derivative is

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) + f(x_0 + h) - 2f(x_0)\right] - \frac{h^2}{12} f^{(4)}(\xi).$$

   Exercise. Use Taylor's expansion to derive this formula.
   Exercise. If $f(0) = 0.4$, $f(-0.1) = 0.7$, and $f(0.1) = 0.2$, find the approximation of $f''(0)$.

Numerical Integration

1. Some concepts:
   (1) What is a basic, and a composite, numerical integration rule?
   (2) What is the degree of precision for a numerical integration rule?
   Exercise. Find the numbers a, b, and c so that the numerical quadrature

$$\int_{-2}^{2} f(x)\, dx \approx a f(-1) + b f(0) + c f(1)$$

   has the degree of precision as high as possible. What is the degree of precision of this quadrature rule with your values of a, b, and c?

2. The basic mid-point rectangle formula is

$$\int_\alpha^\beta f(x)\,dx = f\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha) + \frac{1}{24}f''(\xi)(\beta-\alpha)^3.$$

Exercise. Derive this formula using Taylor's expansion.
Exercise. What is the degree of precision of this formula?
Exercise. Let $a < b$. Let $n \ge 2$ be an integer and $h = (b-a)/n$. Assume f'' is continuous on $[a,b]$. Drive the error term for the composite mid-point rectangle rule

$$\int_a^b f(x)\,dx \approx h\sum_{j=1}^n f\left(a + (j-1/2)h\right).$$

3. The basic and composite trapezoid rules are:

$$\int_\alpha^\beta f(x)\,dx = \frac{1}{2}\left[f(\alpha)+f(\beta)\right](\beta-\alpha) - \frac{1}{12}f''(\xi)(\beta-\alpha)^3;$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a)+f(b)+2\sum_{j=1}^{n-1}f(x_j)\right] - \frac{b-a}{12}f''(\xi)h^2,$$

where $n \ge 2$ is an integer, $h = (b-a)/n$, and $x_j = a + jh$ $(j = 0, 1, \ldots, n)$.
Exercise. What is the geometrical meaning of this formula?

4. The basic and composite Simpson's rules are

$$\int_\alpha^\beta f(x)\,dx = \frac{\beta-\alpha}{6}\left[f(\alpha)+4f\left(\frac{\alpha+\beta}{2}\right)+f(\beta)\right] - \frac{1}{2880}f^{(4)}(\xi)(\beta-\alpha)^5,$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a)+f(b)+2\sum_{j=1}^{n/2-1}f(x_{2j})+4\sum_{j=1}^{n/2}f(x_{2j-1})\right] - \frac{b-a}{180}f^{(4)}(\xi)h^4,$$

where $n \ge 2$ is an even integer, $h = (b-a)/n$, and $x_j = a + jh$ $(j = 0, 1, \ldots, n)$.
Exercise. What is the degree of precision of this formula?
Exercise. Write down the pseudocode of this method.
Exercise. If $f(0) = 1$, $f(0.1) = 1.2$, $f(0.2) = 1.4$, $f(0.3) = 2$, and $f(0.4) = 1.8$. Use the composite Simpson's rule (with $n = 2$) to approximate the integral of f over $[0, 0.4]$.

5. If $p_n(x)$ is the Lagrange interpolation of $f(x)$ at $x_0, x_1, \ldots, x_n$ in $[a,b]$, then the (closed) Newton–Cotes formula is constructed so that the integral of f over $[a,b]$ is approximated by the integral of $p_n(x)$ over $[a,b]$. Let $n \ge 1$ be an integer, $h = (b-a)/n$, $x_i = a + ih$ $(i = 0, 1, \ldots, n)$, and $L_i(x)$ $(i = 0, 1, \ldots)$ the basic Lagrange polynomials associated with $x_0, x_1, \ldots, x_n$. Then the basic (closed) Newton–Cotes formula is

$$\int_a^b f(x)\,dx = \sum_{i=0}^n a_i f(x_i) + e_n \qquad \text{where} \quad e_n = \begin{cases} c_n h^{n+3} f^{(n+2)}(\xi) & \text{if } n \text{ is even,} \\ d_n h^{n+2} f^{(n+1)}(\xi) & \text{if } n \text{ is odd,} \end{cases}$$

where c_n and d_n are some constants (cf. Theorem 4.2 on page 196), and $a_i = \displaystyle\int_a^b L_i(x)\,dx$.
Exercise. Why this formula is exact for a polynomial of degree $\le n$?
Exercise. Why is the degree of precision of a Newton–Cotes formula?
Exercise. Let $n = 2$ and derive Simpson's rule for the integration of $f(x)$ over $[-1, 1]$ (without the error term).

6. What is an open Newton–Cotes formula?
7. For any $n \geq 1$, the n-point Gaussian quadrature (with error) is

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where $x_1, \ldots, x_n \in (-1, 1)$ are the n distinct roots of the Legendre polynomial $P_n(x)$ and w_i are the integrals of the basic Lagrange interpolation polynomials $L_i(x)$ associated with these roots. Note that $\{P_k(x)\}_{k=0}^{\infty}$ are orthogonal polynomials.
Exercise. Prove this formula has degree of precision $2n - 1$. (See Theorem 4.7 on page 231.)
Exercise. Use the composite 2-point Gaussian quadrature to approximate the integral of $f(x) = x^4$ on $[0, 2]$ with 2 subintervals.

Richardson's Extrapolation

1. Suppose $M = N(h) + K_2h^2 + K_4h^4 + \cdots$. If you compute $N(h)$ and also $N(h/2)$, then $N(h/2)$ should approximate M better than $N(h)$. You may want to continue with computing $N(h/4)$. But, instead, you can use Richardson's extrapolation to compute $\tilde{N}(h) := [4N(h/2) - N(h)]/3$.
Exercise. Expand $M = \tilde{N}(h) + \tilde{K}_4h^4 + \cdots$. What is $\tilde{K}_4$ in terms of K_4?
Exercise. What is Richardson's extrapolation formula using $N(h)$ and $N(h/3)$?
2. **Exercise.** Show that one step of Richardson's extrapolation of the trapezoidal rule of numerical integration is exactly Simpson's rule.

Local Truncation Errors in Solving Initial-Value Problems of ODE

We consider the initial-value problem for ODE: $y' = f(t, y)$ $(a \leq t \leq b)$ and $y(a) = \alpha$. Let $N \geq 1$ be an integer, $h = (b - a)/N$, and $t_i = a + ih$ $(i = 0, 1, \ldots, N)$. A general numerical method has often the form

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + h\phi(t_i, w_i), \quad i = 0, 1, \ldots, N - 1. \end{aligned}$$

The truncation error for such a method is defined by

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \quad i = 0, 1, \ldots, N - 1.$$

Note that the truncation error is defined using the exact solution $y = y(t)$.

Taylor's Methods for Solving Initial-Value Problems of ODE

1. Euler's method is defined by

$$\begin{aligned} &w_0 = \alpha, \\ &w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \ldots, N - 1. \end{aligned}$$

If L is the Lipschitz constant for $f(t, y)$ in y and $|y''(t)| \leq M$ $(a \leq t \leq b)$, then error bound is

$$|y_i - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right], \qquad i = 0, 1, \ldots, N.$$

Exercise. If $f(t, y) = y + t$ and $y(0) = 0$, find approximations of $y(0.1)$ and $y(0.2)$.
Exercise. Prove the error bound formula.

2. The general Taylor method of order n is given by

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad i = 0, 1, \ldots, N-1,$$

where
$$T^{(n)}(t, w) = f(t, w) + \frac{h}{2!} f'(t, w) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t, w).$$

It is derived using Taylor's expansion

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i) + \frac{h^2}{2!} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)).$$

Exercise. What is the local truncation error for this method?
Exercise. If $f(t, y) = t^2 + \sin y$, what is $f'(t, y)$?

Runge-Kutta Methods for Solving Initial-Value Problems of ODE

1. The midpoint method is a special Runge-Kutta method of order 2. It is given by

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hf\left(t_i + h/2, w_i + (h/2)f(t_i, w_i)\right), \quad i = 0, 1, \ldots, N-1.$$

Exercise. Let $f(t, y) = t + y$ $(0 \le t \le 3)$, $\alpha = 0$, $h = 1$. Compute w_0, w_1 and w_2.
Exercise. What is the order of the local truncation error of this method?

2. The modified Euler's method is also a Runge-Kutta method of order 2. It is given by

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right], \quad i = 0, 1, \ldots, N-1.$$

Exercise. Let $f(t, y) = t + y$ $(0 \le t \le 3)$, $\alpha = 0$, $h = 1$. Compute w_0, w_1 and w_2.
Exercise. What is the order of the local truncation error of this method?

3. A commonly used Runge-Kutta method of order 4 is given by

$$\begin{aligned}
w_0 &= \alpha, \\
k_1 &= hf(t_i, w_i), \\
k_2 &= hf(t_i + h/2, w_i + k_1/2), \\
k_3 &= hf(t_i + h/2, w_i + k_2/2), \\
k_4 &= hf(t_{i+1}, w_i + k_3), \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad i = 0, 1, \ldots, N-1.
\end{aligned}$$

Exercise. Write a pseudocode for this method.
Exercise. Let $f(t, y) = t + y$ $(0 \le t \le 3)$, $\alpha = 0$, $h = 1$. Compute w_0 and w_1.

Multistep Methods for Solving Initial-Value Problems of ODE

1. Examples. An explicit fourth-order Adams–Bashforth method (cf. (5.25) on page 303).
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