Math 18, Linear Algebra, Lecture C00, Spring 2017
Review and Practice Problems for Final Exam

1. The augmented matrix of a linear system has been transformed by row operations into $\begin{bmatrix} 1 & 5 & 2 & -1 \\ 0 & 2 & -4 & 8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Determine if the system is consistent.

2. Choose h and k so that the system of equations $x_1 + hx_2 = 2$ and $4x_1 + 8x_2 = k$ has (a) no solution, (b) one and only one solution, and (c) infinitely many solutions.

3. Solve the linear system $x_1 - 2x_3 = 9$ and $x_2 + x_3 = 3$; and write down the solution set.

4. Find the solution of the system $x_1 + 5x_2 - 3x_3 = 0$, $x_1 - x_2 + x_3 = -2$.

5. Find a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

6. Is the vector $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$? (That a vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ means that $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some numbers c_1, c_2, and c_3. So, you need to solve for these numbers.)

7. Can three linearly dependent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$? What about three linearly independent vectors in $\mathbb{R}^3$?

8. Do $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ span $\mathbb{R}^3$?

9. For what values of h will $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ be in the span of $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$?

10. Let A be a 10×20 matrix. Suppose the first 5 column vectors are linearly dependent. Are all column vectors of A also linearly dependent?

11. If a matrix has more columns than rows, then the columns of this matrix are always linearly dependent. Is this true?

12. Find h so that the vectors $\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$ are linearly dependent.

13. Three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent if and only if one of them is a linear combination of the other two. Explain why.

14. Define a transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $T(\mathbf{x}) = (2x_1 + 3, 3x_2 - x_1)$. Is T linear? If so, find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in R^2.

15. Define a transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $T(\mathbf{x}) = (2x_1x_2, 3x_2 - x_1)$. Is T linear? If so, find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in R^2.

16. Define a transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $T(\mathbf{x}) = (2x_1 - x_2, 3x_2 - x_1)$. Is T linear? If so, find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in R^2.

17. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two vectors in $\mathbb{R}^3$ such that $T(\mathbf{u}_1) = \mathbf{0}$ and $T(\mathbf{u}_2) = \mathbf{0}$. Let $\mathbf{u}$ be in the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$. Show that $T(\mathbf{u}) = \mathbf{0}$.

18. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose T transforms $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively. Find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

19. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector with respect to the x_2-axis and then expands by a factor of 2 in the x_1-direction. Find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

20. Let $A = \begin{bmatrix} 5 & -4 & 5 \\ 1 & -2 & -1 \\ -1 & 5 & 6 \end{bmatrix}$. Is the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? onto? Is the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ in the range of T?

21. Can a linear transformation from R^1 to $\mathbb{R}^2$ be onto? Why?

22. Let $A = \begin{bmatrix} 0 & -1 & 3 \\ 2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -1 \\ 1 & 3 \\ -3 & -4 \end{bmatrix}$. Find out $(AB)^T$ and $-3B + 2C$.

23. Find two 2×2 matrices A and B such that $AB \neq BA$.

24. Find two 2×2 matrices A and B such that $AB = 0$ but $A \neq 0$ and $B \neq 0$.

25. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

26. Let $A = \begin{bmatrix} -5 & -1 & 9 \\ 3 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$. Is A invertible? If so, find its inverse.

27. Find three 3×3 elementary matrices of three different types. Calculate their inverse matrices.

28. Invertible matrix: definition. If $AB = BA = I$, then $B = A^{-1}$. How to check if an $n \times n$ matrix A is invertible?

    (a) Echelon form: n pivots; nonzero diagonal entries;
    (b) All columns are linearly independent;
    (c) All columns span $\mathbb{R}^n$;
    (d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;
    (e) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$;
    (f) $\det A \neq 0$;
    (g) A^T is invertible;
    (h) The number 0 is not an eigenvalue of A.

29. Let T be a one-to-one linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. Let A be the matrix associated with T. Is A invertible?

30. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. Find out the LU factorization of A and use this to solve $A\mathbf{x} = \mathbf{b}$ by solving two triangular systems.

31. Find all the co-factors, the adjugate matrix, and the determinant of the matrix $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & -4 & 1 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 \end{bmatrix}$.

32. True or false:

    (a) If a square matrix B is an echelon form of matrix A then $\det A = \det B$;
    (b) The determinant of any elementary matrix is 1;
    (c) $\det(A + B) = \det A + \det B$, $\det(-A) = -\det A$, and $\det(AB) = (\det A)(\det B)$;
    (d) $\det A^T = \det A$ and $\det A^{-1} = (\det A)^{-1}$;
    (e) A square matrix is invertible if and only if its determinant is nonzero.

33. Calculate the determinant of each of the following matrices:

$$\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 0 \\ -1 & 3 & 12 & 0 \\ 1 & 2 & -4 & 1 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & 5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}.$$

34. Show that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (a - b)(b - c)(c - a)$.

35. What is Cramer's rule? Let A be a 3×3 matrix and assume $\det A = 2$, the co-factors $C_{11} = 2$, $C_{12} = -4$, $C_{13} = 6$. What is the solution to $A\mathbf{x} = \mathbf{e}_1$ with $\mathbf{e}_1 = (1, 0, 0)^T$?

36. Since $A^{-1} = \frac{1}{\det A}\,\mathrm{adj\,A}$, we have $A\,\mathrm{adj}\,A = (\det A)I$. Correct?

37. Find the area of the parallelogram whose vertices are $(0,-2),(6,-1),(-3,1),(3,2)$.

38. Find the volume of the parallelepiped with one vertex at $(0,0,0)$ and its three adjacent vertices $(1,4,0)$, $(-1,2,-1)$, and $(-2,-5,2)$.

39. Show that the area of the triangle with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ is the absolute value of $\frac{1}{2}\det\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}$.

40. Determine if H is a subspace of $\mathbb{R}^3$:
    (a) H consists of all the vectors in $\mathbb{R}^3$ with the product of all the components equal to 0;
    (b) H consists of all the vectors in $\mathbb{R}^3$ with the first component equal to 1;
    (c) H consists of all the vectors in $\mathbb{R}^3$ with the first component equal to 0;
    (d) H consists of all the vectors $\begin{bmatrix}2a-b\\ 2b-c\\ 2c-a\end{bmatrix}$ with a,b,c all real numbers.

41. Is the set of all 2×2 diagonal matrices a subspace of the space of all 2×2 matrices?

42. Determine if $\begin{bmatrix}0\\1\\1\\-2\end{bmatrix}$ is in the span by $\begin{bmatrix}1\\1\\-2\\0\end{bmatrix}$, $\begin{bmatrix}-2\\0\\2\\2\end{bmatrix}$ and $\begin{bmatrix}1\\4\\-4\\1\end{bmatrix}$.

43. Determine if $\begin{bmatrix}2\\1\end{bmatrix}$ is in the column space and null space of $\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$.

44. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors in a vector space. Show that any three vectors in $\mathrm{Span}\,\{\mathbf{v}_1,\mathbf{v}_2\}$ must be linearly dependent.

45. Show that $p_0(t)=1$, $p_1(t)=1+t$, and $p_2(t)=1+t+t^2$ form a basis of the vector space $\mathbb{P}_2$. Find the coordinates of $p(t)=2-t+3t^2$ with respect to this basis.

46. Find a basis for the column space, the null space, and row space of $\begin{bmatrix}1 & -2\\ -2 & 7\\ 3 & -9\end{bmatrix}$ and $\begin{bmatrix}1 & 0 & -1 & 5\\ 0 & 1 & 0 & 2\\ 2 & 0 & 2 & -1\end{bmatrix}$. Find also the rank of each of these two matrices.

47. True or false:
    (a) If B is an echelon form of A, then A and B have the same null space, column space, and row space;
    (b) The row space of A is the column space of A^T;

(c) If the null space of a square matrix A is $\{\mathbf{0}\}$ then A is invertible;
(d) If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^3$ then the rank of the matrix $\mathbf{u}\mathbf{v}^T$ is always 0 or 1.

48. What is the dimension of $\mathbb{R}^4$, $\mathbb{P}_4$, the space of all 4×4 matrices?

49. If V is a vector space and $\dim V = n$ then $n+1$ vectors in V must be linearly dependent. Correct? Why?

50. True or false:
    (a) $\operatorname{rank} A + \operatorname{rank} B = \operatorname{rank}(A + B)$;
    (b) $\operatorname{rank}(AB) = (\operatorname{rank} A)(\operatorname{rank} B)$;
    (c) $\dim \operatorname{Col} A + \dim \operatorname{Nul} A =$ number of columns of A.

51. True or false: (Justify your answers.)
    (a) If λ is an eigenvalue of A. Then λ^2 is an eigenvalue of A^2;
    (b) If λ is an eigenvalue of A then $\lambda - 7$ is an eigenvalue of the matrix $A - 7I$; (I is the identity matrix.)
    (c) An eigenvalue of a matrix can never be 0, and an eigenvector of a matrix can never be the zero vector;
    (d) The sum of all eigenvalues of a matrix equals the sum of all diagonal entries of the matrix;
    (e) The product of all eigenvalues of a 3×3 equals the determinant of the matrix. What about a 2×2 matrix?
    (f) If two matrices have same eigenvalues then either both of them are diagonalizable or both of them are not;
    (g) Any invertible matrix is diagonalizable;
    (h) The inverse of an invertible and diagonalizable matrix is diagonalizable.
    (i) Any upper triangular square matrix is diagonalizable;
    (j) The product of two diagonalizable matrices is still diagonalizable.

52. Explain why any three different eigenvectors of a matrix A corresponding to three different eigenvalues must be linearly independent. (The statement is true for any number, not necessary three.)

53. Write down the characteristic equation, and then find all the eigenvalues and eigenvectors of each of the following matrices: $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$ and $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$.

54. Two matrices A and B (of same size) are similar, if $A = PBP^{-1}$ for some invertible matrix P (cf. Section 5.2). Why similar matrices have the same characteristic equations, and hence have the same eigenvalues, and also same determinants?

55. Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. Find all eigenvalues and the corresponding eigenvectors of A. Then determine if A is diagonalizable. If so, diagonalize it (i.e., find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$). Finally, compute A^6.

56. The matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ has one eigenvalue equal to 2. Diagonalize A.

57. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

    (a) Compute $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u}^T\mathbf{v}$, the length of $\mathbf{u}$, and the distance between $\mathbf{u}$ and $\mathbf{v}$.
    (b) Find all the vectors $\mathbf{x}$ that are orthogonal to $\mathbf{u}$.
    (c) Find all the vectors $\mathbf{x}$ that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
    (d) Find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.
    (e) Find the orthogonal projection of $\mathbf{y}$ onto the subspace spanned by $\mathbf{u}$ and $\mathbf{v}$.

58. Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal. Show that $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

59. Prove that nonzero and mutually orthogonal vectors are linearly independent.

60. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$. Let $\mathbf{y} \in W$. Show that

$$\mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \ldots \left(\frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\right)\mathbf{u}_p.$$

61. What do we mean by orthonormal vectors?

62. Let U be an orthogonal matrix and $\mathbf{u}$ a unit vector. Show that $U\mathbf{u}$ is also a unit vector. Show that the inner product of $U\mathbf{x}$ and $U\mathbf{y}$ for any $\mathbf{x}$ and $\mathbf{y}$ is the same as that of $\mathbf{x}$ and $\mathbf{y}$.

63. Show that the determinant of an orthogonal matrix is 1 or -1.

64. Do all the elementary row reductions preserve the orthogonality of a matrix? Why?

65. The matrix $\begin{bmatrix} 1/\sqrt{18} & 1/\sqrt{2} & -2/3 \\ 4/\sqrt{18} & 0 & 1/3 \\ 1/\sqrt{18} & -1/\sqrt{2} & -2/3 \end{bmatrix}$ is an orthogonal matrix. (Verify that!) Find its inverse. (It's simple to find the inverse matrix of an orthogonal matrix. You don't need to use the elementary row reduction.)

66. Let $\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. Find the orthogonal projection of $\mathbf{y}$ onto W and the distance from $\mathbf{y}$ to W.

67. What is the Gram–Schmidt orthogonalization process? Apply this process to $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$ to get three orthogonal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

68. Work out Problems 2 and 9 of Section 6.4.