Solution to Practice Problems for Final Exam

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#51. (a) Yes. $A\vec{u}=\lambda\vec{u}\ (\vec{u}\neq\vec{0}) \Rightarrow A(A\vec{u})=A(\lambda\vec{u})$
$\Rightarrow A^2\vec{u}=\lambda A\vec{u}=\lambda\lambda\vec{u}=\lambda^2\vec{u},\ \vec{u}\neq 0.$

(b) Yes. $A\vec{u}=\lambda\vec{u}\ (\vec{u}\neq\vec{0}) \Rightarrow (A-7I)\vec{u}=A\vec{u}-7\vec{u}$
$=\lambda\vec{u}-7\vec{u}=(\lambda-7)\vec{u}\ \ (\vec{u}\neq\vec{0})$

(c) 0 can be eigenvalue of A. e.g., $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

In fact, 0 is an eigenvalue of $A \Leftrightarrow \det A=0$
[since in this case $A\vec{x}=\vec{0}$ has a nonzero solution $\vec{x}$. So $A\vec{x}=0\vec{x}$.]

Eigenvector is never $\vec{0}$. — by definition.

(d) Yes. The pf is not required. But, if you want to know why, here is reason:

$\det(A-\lambda I)=(-1)^n\lambda^n+(-1)^{n-1}(a_{11}+\cdots+a_{nn})\lambda^{n-1}+\cdots$
$=(-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

Compare the coefficients of λ^{n-1}!

(e) Yes. $\det(A-\lambda I)=(-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$
where $\lambda_1,\lambda_2,\lambda_3$ are the three eigenvalues of A. Let $\lambda=0$. $\det A=(-1)^3(-\lambda_1)(-\lambda_2)(-\lambda_3)=\lambda_1\lambda_2\lambda_3$.

If A is 2×2, then $\det A=\lambda_1\lambda_2$. Similar!
True for a general $n\times n$ matrix.

(f) False. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

(g) False. $\begin{bmatrix}1&1\\0&1\end{bmatrix}$

(h) True. $A=PDP^{-1}\Rightarrow A^{-1}=PD^{-1}P^{-1}$.

(i) False. $\begin{bmatrix}1&1\\0&1\end{bmatrix}$

(j) False. $\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$, $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ product: $\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$.

#52. Let $A\vec{u}_1 = \lambda_1\vec{u}_1$, $A\vec{u}_2 = \lambda_2\vec{u}_2$, $A\vec{u}_3 = \lambda_3\vec{u}_3$, $\lambda_1, \lambda_2, \lambda_3$: distinct. $\vec{u}_1, \vec{u}_2, \vec{u}_3$: nonzero. Assume $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$. Then $\lambda_1(\cdots) = \lambda_1 \cdot \vec{0} = \vec{0}$

i.e. $\lambda_1 c_1\vec{u}_1 + \lambda_1 c_2\vec{u}_2 + \lambda_1 c_3\vec{u}_3 = \vec{0}$ ---- ①

Also, $A(\cdots) = A\vec{0} = \vec{0}$. i.e. $c_1 A\vec{u}_1 + c_2 A\vec{u}_2 + c_3 A\vec{u}_3 = \vec{0}$

So, $c_1\lambda_1\vec{u}_1 + c_2\lambda_2\vec{u}_2 + c_3\lambda_3\vec{u}_3 = \vec{0}$ ---- ②

Eqns ①. ② $\Rightarrow$ $c_2(\lambda_1 - \lambda_2)\vec{u}_2 + c_3(\lambda_1 - \lambda_3)\vec{u}_3 = \vec{0}$ ··· ③

$\lambda_2(\cdots) = \lambda_1\vec{0} = \vec{0} \Rightarrow c_2\lambda_2(\lambda_1 - \lambda_2)\vec{u}_2 + c_3\lambda_2(\lambda_1 - \lambda_3)\vec{u}_3 = \vec{0}$

$A(\cdots) = A\vec{0} = \vec{0} \Rightarrow c_2(\lambda_1 - \lambda_2)A\vec{u}_2 + c_3(\lambda_1 - \lambda_3)A\vec{u}_3 = \vec{0}$

$\Rightarrow c_2\lambda_2(\lambda_1 - \lambda_2)\vec{u}_2 + c_3\lambda_3(\lambda_1 - \lambda_3)\vec{u}_3 = \vec{0}$

Hence, $c_3(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)\vec{u}_3 = \vec{0}$ $\vec{u}_3 \neq 0$. $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3 \Rightarrow c_3 = 0$. Eq ③ $\Rightarrow c_2 = 0$. So. $c_1 = 0$ as well. [or use induction.]

#53 $A = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$ $\det(A - \lambda I) = 0$ $\begin{vmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = 0$

$(9-\lambda)(5-\lambda) + 4 = 0$ $\underline{\lambda^2 - 14\lambda + 49 = 0}$ the charc. eq.

$(\lambda - 7)^2 = 0$ $\lambda_1 = \lambda_2 = 7$ (all the eigenvalues)

$\begin{bmatrix} 9-7 & -2 \\ 2 & 5-7 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 - x_2 = 0$ $\vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

All eigenvectors: $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(x_2 \neq 0)$

$A = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$ $\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0$

$(3-\lambda)[(9-\lambda)(6-\lambda) - 4] = 0$

$(\lambda^2 - 15\lambda + 50)(\lambda - 3) = 0$. char. eq.

$\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_3 = 10$.

$\lambda_1 = 3$ $A - 3I = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 6 & 0 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$x_1=0,\ x_2=0.$ $\vec{x} = x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$ $(x_3 \neq 0)$ — all eigenvectors corresp. to $\lambda_1=3$

$\lambda_2=5$ $A-5I=\begin{bmatrix}1 & -2 & 0\\ -2 & 4 & 0\\ 5 & 8 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0\\ 0 & 18 & -2\\ 0 & 0 & 0\end{bmatrix}$

$\sim \begin{bmatrix}1 & -2 & 0\\ 0 & -9 & 1\\ 0 & 0 & 0\end{bmatrix}$ $\vec{x} = \begin{bmatrix}2x_2\\ x_2\\ 9x_2\end{bmatrix} = x_2\begin{bmatrix}2\\1\\9\end{bmatrix}$ $(x_2\neq 0)$ all eigenvectors for $\lambda_2=5$.

$\lambda_3=10$ $A-10I=\begin{bmatrix}-4 & -2 & 0\\ -2 & -1 & 0\\ 5 & 8 & -7\end{bmatrix} \sim \begin{bmatrix}2 & 1 & 0\\ 5 & 8 & -7\\ 0 & 0 & 0\end{bmatrix}$

$\sim \begin{bmatrix}2 & 1 & 0\\ 10 & 16 & -14\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}2 & 1 & 0\\ 0 & 11 & -14\\ 0 & 0 & 0\end{bmatrix}$

$\sim \begin{bmatrix}2 & 1 & 0\\ 0 & 1 & -14/11\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}2 & 0 & 14/11\\ 0 & 1 & -14/11\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 7/11\\ 0 & 1 & -14/11\\ 0 & 0 & 0\end{bmatrix}$

$\vec{x} = x_3\begin{bmatrix}-7/11\\ 14/11\\ 1\end{bmatrix} = \frac{x_3}{11}\begin{bmatrix}-7\\14\\11\end{bmatrix}$ $(x_3\neq 0)$ all eigenvectors for $\lambda_3=10$.

#54. $A=PBP^{-1}$. $\det(A-\lambda I) = \det(PBP^{-1}-\lambda I)$
$= \det(PBP^{-1} - P\lambda I P^{-1}) = \det(P(B-\lambda I)P^{-1})$
$= \det P \det(B-\lambda I)\det P^{-1} = \det(B-\lambda I)$
since $\det P = \det P^{-1}$.
So, the char. eq. of A is the same as that of B. Hence, A, B have the same eigenvalues, since eigenvalues of roots of char. eq. $\det(A-\lambda I) = \det(B-\lambda I)$
for $\lambda=0$: $\det A = \det B$.

#55 $A=\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix}4-\lambda & -3\\ 2 & -1-\lambda\end{vmatrix} = (\lambda-4)(\lambda+1)+6=0$
$\lambda^2-3\lambda+2=0$ $\lambda_1=1,\ \lambda_2=2$: all eigenvalues.

$\lambda_1 = 1.\quad A - \lambda_1 I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\quad \vec{u}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\alpha \neq 0)$

$\lambda_2 = 2\quad A - \lambda_2 I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}\quad \vec{u}_2 = \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix} \alpha \neq 0.$

$\vec{u}_1, \vec{u}_2$: L.I. So, A is diagonalizable.

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $P = [\vec{u}_1\ \vec{u}_2] = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\quad P^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

$A = PDP^{-1}$. check $AP = PD$.

$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 190 & -189 \\ 126 & -125 \end{bmatrix}$

#56. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & 4-\lambda & 1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0\quad (2-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$

$+ \begin{vmatrix} 1 & 1 \\ -1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 4-\lambda \\ -1 & -1 \end{vmatrix} = 0$

$(2-\lambda)[(\lambda-4)(\lambda-2)+1] + (2-\lambda) + 1 - (-1 + 4 - \lambda) = 0$

$(2-\lambda)[\lambda^2 - 6\lambda + 9] + (2-\lambda) \cancel{+2} (2-\lambda) = 0$

$\cancel{(\lambda-2)^2(\lambda+1)(\lambda-7) = 0 \quad \lambda_1 = \lambda_2 = \lambda_3 = 2}$

$(2-\lambda)(\lambda-3)^2 = 0\qquad \lambda_1 = 2.\ \lambda_2 = \lambda_3 = 3.$

$\lambda_1 = 2.\quad A - \lambda_1 I = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\qquad \vec{u}_1 = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\quad \alpha \neq 0.$

$\lambda_2 = \lambda_3 = 3\quad A - \lambda_2 I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{= u_2} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{= u_3}$

$A = PDP^{-1}\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.\quad P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

#57 (a) $\vec{u}\cdot\vec{v} = 2\cdot 1 + (-1)(-2) + 0\cdot 1 = 4.$

$\vec{u}^T\vec{v} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 4.$

length: $\|\vec{u}\| = \sqrt{4+1} = \sqrt{5}$

distance: $\|\vec{u}-\vec{v}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}.$

(b) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$ $\vec{x}\cdot\vec{u} = \vec{0} \Longrightarrow 2x_1 - x_2 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ 2x_1 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ x_1, x_3: free.

(c) $\vec{x}\cdot\vec{u} = 0,\ \vec{x}\cdot\vec{v} = 0 \Longrightarrow$ $2x_1 - x_2 = 0$
$x_1 - 2x_2 + x_3 = 0$

$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2/3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -2/3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} \frac{1}{3}x_3 \\ \frac{2}{3}x_3 \\ x_3 \end{bmatrix} = \frac{x_3}{3}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ x_3: free.

(d) $\text{Proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{4}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \dfrac{2}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

(e) Let $W = \text{span}\{\vec{u}, \vec{v}\}.$

~~Then~~ $\cancel{\text{Proj}_W\vec{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \dfrac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}}$

Let $\vec{z} = \text{Proj}_W\vec{y}$, the orthog. projection of $\vec{y}$ onto W. Then

$(\vec{z}-\vec{y})\cdot\vec{u} = 0$. $(\vec{z}-\vec{y})\cdot\vec{v} = 0$

$\vec{z} = a\vec{u} + b\vec{v}.$

$\vec{z}\cdot\vec{u} = \vec{y}\cdot\vec{u} \Rightarrow a\,\vec{u}\cdot\vec{u} + b\,\vec{u}\cdot\vec{v} = \vec{y}\cdot\vec{u} \Rightarrow 5a + 4b = -2$

$\vec{z}\cdot\vec{v} = \vec{y}\cdot\vec{v} \Rightarrow a\,\vec{u}\cdot\vec{v} + b\,\vec{v}\cdot\vec{v} = \vec{y}\cdot\vec{v}$ $4a + 6b = -1$

$a = -4/7,\ b = 3/14$

$\vec{z} = -\frac{4}{7}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{14}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix}.$

#58. $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})^T(\vec{u}+\vec{v}) = \vec{u}^T(\vec{u}+\vec{v}) + \vec{v}^T(\vec{u}+\vec{v})$

$= \vec{u}^T\vec{u} + \underbrace{\vec{u}^T\vec{v}}_{(\vec{u}\cdot\vec{v}=0)} + \underbrace{\vec{v}^T\vec{u}}_{(\vec{v}\cdot\vec{u}=0)} + \vec{v}^T\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2$

#59. Let $\vec{u}_1, \dots, \vec{u}_k$ be nonzero, mutually orthog vectors in $\mathbb{R}^n$. Let $c_1\vec{u}_1 + \dots + c_k\vec{u}_k = \vec{0}$

Then $\vec{u}_1 \cdot (\dots) = \vec{u}_1 \cdot \vec{0} = 0$

i.e., $c_1 \vec{u}_1\cdot\vec{u}_1 + c_2 \underbrace{\vec{u}_1\cdot\vec{u}_2}_{=0} + \dots + c_k \underbrace{\vec{u}_1\cdot\vec{u}_k}_{=0} = 0$

$\Longrightarrow c_1\|\vec{u}_1\|^2 = 0$

But $\vec{u}_1 \neq \vec{0}$ so $c_1 = 0$.

Similarly all $c_2 = \dots = c_k = 0$. So $\vec{u}_1, \dots, \vec{u}_k$ are L.I.

#60. Since $\vec{y} \in W$ and $\{\vec{u}_1, \dots \vec{u}_p\}$ is a basis for W, we have $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_p\vec{u}_p$.

$\vec{u}_1 \cdot \vec{y} = \vec{u}_1(\dots) = c_1\vec{u}_1\cdot\vec{u}_1 + c_2\underbrace{\vec{u}_1\cdot\vec{u}_2}_{=0} + \dots + c_p\underbrace{\vec{u}_1\cdot\vec{u}_p}_{=0}$

$\Longrightarrow \vec{u}_1\cdot\vec{y} = c_1\vec{u}_1\cdot\vec{u}_1$.

$c_1 = \vec{y}\cdot\vec{u}_1 / \vec{u}_1\cdot\vec{u}_1$. Similarly $c_2 = \dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}$ ..etc.

#61. orthonormal vectors: orthogonal, and unit vectors (length = 1)

#62. $U^TU = I$. So $U\vec{x}\cdot U\vec{y} = (U\vec{x})^T U\vec{y} = \vec{x}^T U^T U\vec{y}$
$= \vec{x}^T I \vec{y} = \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$.

If $\vec{y} = \vec{x}$ then $U\vec{x}\cdot U\vec{x} = \|U\vec{x}\|^2 = \vec{x}\cdot\vec{x} = \|\vec{x}\|^2$.

#63. If U is an orthog matrix, then $U^TU = I$.

So, $\det(U^TU) = \det I = 1$.

But $\det U^T = \det U$; $\det(U^TU) = \det U^T \det U$

So $(\det U)^2 = 1$ and $\det U = 1$ or -1.

#64. No. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. orthogonal But $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ is not

#65. $\begin{bmatrix} 1/\sqrt{18} & 4/\sqrt{18} & 1/\sqrt{18} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -2/3 & 1/3 & -2/3 \end{bmatrix}$ Just the transpose.

#66. Note that $\vec{u}\cdot\vec{v}=0$. So, the projection is

$$\text{Proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} + \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= \frac{35}{35}\begin{bmatrix}-3\\5\\1\end{bmatrix} + \frac{-28}{14}\begin{bmatrix}-3\\2\\1\end{bmatrix} = \begin{bmatrix}-3\\5\\1\end{bmatrix} - 2\begin{bmatrix}-3\\2\\1\end{bmatrix} = \begin{bmatrix}3\\1\\-1\end{bmatrix}$$

(Compare this with Part (e) of Prob. 57, where $\vec{u}, \vec{v}$ are not orthog.)

#67. $\vec{u}_1, \cdots, \vec{u}_k$: L.I. $\xRightarrow{\text{G-S.}}$ $\vec{v}_1, \cdots, \vec{v}_k$ ① orthogonal

② $\text{span}\{\vec{v}_1\} = \text{span}\{\vec{u}_1\}$

$$\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

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$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = \text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix}1\\2\\-2\end{bmatrix}. \quad \vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{bmatrix}1\\0\\-4\end{bmatrix} - \frac{9}{9}\begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}0\\-2\\-2\end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{u}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \begin{bmatrix}5\\2\\0\end{bmatrix} - \frac{9}{9}\begin{bmatrix}1\\2\\-2\end{bmatrix} - \frac{-4}{8}\begin{bmatrix}0\\-2\\-2\end{bmatrix} = \begin{bmatrix}4\\-1\\1\end{bmatrix}.$$

#68. Apply the formulas in Theorem 11 on page 357.

Answer: #2. $\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}5\\4\\-8\end{bmatrix}$.

#9. $\begin{bmatrix}3\\1\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\3\\3\\-1\end{bmatrix}, \begin{bmatrix}-3\\1\\1\\3\end{bmatrix}$.