Solutions/Hints to Practice Problems for Midterm Exam 1. (Math 18, Spring 2017)

1. $\boxed{\text{Yes}}$ $x_3 = 0,\ x_2 = 4,\ x_1 = -1 - 5\cdot 4 = -21.$

2. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

(a) no sol'n $8-4h=0,\ k-8\neq 0$ i.e. $\boxed{h=2,\ k\neq 8}$

(b) $8-4h\neq 0$ $\boxed{h\neq 2}$ — one and only one sol'n.

(c) infinitely many sol'n: $8-4h=0,\ k-8=0$

$\boxed{h=2,\ k=8}$

3. $\begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{bmatrix}$ $\cancel{\sim \begin{bmatrix} 1 & -2 & 9 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix}}$

$\cancel{x_1 = 5,\ x_2 = -2}$ $x_1 = 9 + 2x_3,\quad x_2 = 3 - x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9+2x_3 \\ 3-x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$ x_3: free variable

4. $\begin{bmatrix} 1 & 5 & -3 & 0 \\ 1 & -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & 0 \\ 0 & -6 & +4 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 5 & -3 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

$x_1 = -\frac{5}{3} - \frac{1}{3}x_3$

$x_2 = \frac{1}{3} + \frac{2}{3}x_3$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{3} \\ +\frac{2}{3} \\ 1 \end{bmatrix}}$$

5. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Idea: $A = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $a\cdot 2 + b(-1) + c\cdot 2 = 0.$

Many choices of a, b, c.

6. $x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}+x_3\begin{bmatrix}5\\-6\\8\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$

$$\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&0&0&0\end{bmatrix}$$

$\boxed{\text{Yes}}$. many soln's of x_1, x_2, x_3.

$x_1 = 2-5x_3$ $\quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\3\\0\end{bmatrix}+x_3\begin{bmatrix}-5\\-4\\1\end{bmatrix}$.

$x_2 = 3-4x_3$

7. No. ~~Yes.~~

8. $$\begin{bmatrix}2&0&3\\-1&1&-2\\3&1&0\end{bmatrix}\sim\begin{bmatrix}-1&1&-2\\2&0&3\\3&1&0\end{bmatrix}\sim\begin{bmatrix}-1&1&-2\\0&2&-1\\0&4&-6\end{bmatrix}$$

$$\sim\begin{bmatrix}-1&1&-2\\0&2&-1\\0&0&-4\end{bmatrix}$$ Yes. since these three vectors in $\mathbb{R}^3$ are L. I.

9. $x_1\begin{bmatrix}-3\\1\\0\end{bmatrix}+x_2\begin{bmatrix}-5\\4\\7\end{bmatrix}+x_3\begin{bmatrix}1\\-1\\-2\end{bmatrix}=\begin{bmatrix}-4\\3\\h\end{bmatrix}$

$$\begin{bmatrix}-3&-5&1&-4\\1&4&-1&3\\0&7&-2&h\end{bmatrix}\sim\begin{bmatrix}1&4&-1&3\\-3&-5&1&-4\\0&7&-2&h\end{bmatrix}$$

$$\sim\begin{bmatrix}1&4&-1&3\\0&7&-2&5\\0&7&-2&h\end{bmatrix}\sim\begin{bmatrix}1&4&-1&3\\0&7&-2&5\\0&0&0&h-5\end{bmatrix}$$

$\boxed{h=5}$

10. Yes. $\quad c_1\vec{u}_1+\cdots+c_5\vec{u}_5=\vec{0}$ $\quad c_1,\cdots,c_5$: not all 0.

$\Rightarrow$ $\quad c_1\vec{u}_1+\cdots+c_5\vec{u}_5+0\vec{u}_6+\cdots+0\vec{u}_{10}=\vec{0}$.

not all 0.

11. Yes. More unknowns, less equations.

12. $\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 1 & 0 \\ -4 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -6 & 8+h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 38+h & 0 \end{bmatrix}$

$38+h=0$ $\boxed{h=-38}$

13. If $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are L.D., then there are numbers c_1, c_2, c_3, not all 0, such that $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$.
Let's say, $c_1 \neq 0$, then $\vec{u}_1 = (-\frac{c_2}{c_1})\vec{u}_2 + (-\frac{c_3}{c_1})\vec{u}_3$.
So, $\vec{u}_1$ is a L.C. of $\vec{u}_2, \vec{u}_3$.
If, say, $\vec{u}_1$ is a L.C. of $\vec{u}_2, \vec{u}_3$, $\vec{u}_1 = a_2\vec{u}_2 + a_3\vec{u}_3$,
then $\vec{u}_1 - a_2\vec{u}_2 - a_3\vec{u}_3 = \vec{0}$. $1, -a_2, -a_3$ are not all 0. So, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are L.D.

14. No $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = (3,0) \neq (0,0)$.

15. No. The term x_1x_2 is not linear in x_1, x_2.
Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c=2$. $c\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
$T(c\vec{x}) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = (2\cdot 2\cdot 2,\ 3\cdot 2-2) = (8,4)$
$cT(\vec{x}) = 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\,(2\cdot 1\cdot 1,\ 3\cdot 1-1) = 2(2,2) = (4,4)$
$T(c\vec{x}) \neq cT(\vec{x})$. So, not linear.

16. Yes. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$.

17. If $\vec{u} = a\vec{u}_1 + b\vec{u}_2$ is in the span of $\{\vec{u}_1, \vec{u}_2\}$,
then $T(\vec{u}) = aT(\vec{u}_1) + bT(\vec{u}_2) = a\vec{0} + b\vec{0} = \vec{0}$.

18. $x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-1\end{bmatrix} = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{=\vec{e}_1} \Rightarrow x_1 = \frac{1}{2},\ x_2 = \frac{1}{2}$

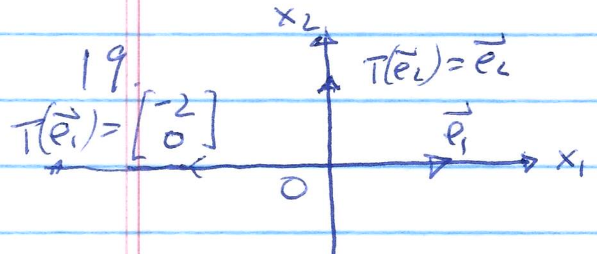
$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}3\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}\frac{5}{2}\\\frac{1}{2}\end{bmatrix}$

$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-1\end{bmatrix} = \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{=\vec{e}_2}\quad x_1 = \frac{1}{2},\ x_2 = -\frac{1}{2}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}3\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$

So, $A = [\,T(\vec{e}_1),\ T(\vec{e}_2)\,] = \begin{bmatrix}\frac{5}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$.

19.



$A = \begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$

20. $A\vec{x} = \vec{0}$ $\begin{bmatrix}5 & -4 & 5 & 0\\ 1 & -2 & -1 & 0\\ -1 & 5 & 6 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1 & 0\\ 5 & -4 & 5 & 0\\ -1 & 5 & 6 & 0\end{bmatrix}$

$\sim \begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 6 & 10 & 0\\ 0 & 3 & 5 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 6 & 10 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ infinitely many sol'ns. (nonzero sol'n)

Not one-to-one.

$A\vec{x} = \vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ $\begin{bmatrix}5 & -4 & 5 & b_1\\ 1 & -2 & -1 & b_2\\ -1 & 5 & 6 & b_3\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1 & b_2\\ 5 & -4 & 5 & b_1\\ -1 & 5 & 6 & b_3\end{bmatrix}$

$\sim \begin{bmatrix}1 & -2 & -1 & b_2\\ 0 & 6 & 10 & 5b_2+b_1\\ 0 & 3 & 5 & b_2+b_3\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1 & b_2\\ 0 & 3 & 5 & b_2+b_3\\ 0 & 6 & 10 & 5b_2+b_1\end{bmatrix}$

$\sim \begin{bmatrix}1 & -2 & -1 & b_2\\ 0 & 3 & 5 & b_2+b_3\\ 0 & 0 & 0 & 3b_2+b_1-b_3\end{bmatrix}$ No sol'n if $3b_2+b_1-b_3 \neq 0$.

So, not onto.

$b_1 = 1,\ b_2 = -1,\ b_3 = 0.\quad 3b_2+b_1-b_3 = -3+1 = -2 \neq 0.$

So, not in the range of T.

21. No. $T(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}[x] = \begin{bmatrix} a_1 x \\ a_2 x \end{bmatrix}$

$\frac{a_1 x}{a_2 x} = \frac{a_1}{a_2}$ fixed, but $\mathbb{R}^2$-vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

will have $\frac{2}{1}$, $\frac{3}{1}$ different ratios.

22. $(AB)^T = B^T A^T = \begin{bmatrix} -5 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -12 & -3 \\ -3 & 5 & 1 \end{bmatrix}$

$-3B + 2C = -3\begin{bmatrix} -5 & 1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 4 & -1 \\ 1 & 3 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 23 & -5 \\ 2 & -3 \\ -9 & -8 \end{bmatrix}$

23 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

24. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, $A \neq 0$, $B \neq 0$

The 2x2 zero matrix.

25. $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

26. $[A \ I] = \begin{bmatrix} -5 & -1 & 9 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ -5 & -1 & 9 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & -1 & 14 & 1 & 0 & 5 \\ 0 & 1 & -8 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -8 & 0 & 1 & -3 \\ 0 & -1 & 14 & 1 & 0 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -8 & 0 & 1 & -3 \\ 0 & 0 & 6 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -8 & 0 & 1 & -3 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & \frac{7}{3} & -1/3 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$ $A^{-1} = \begin{bmatrix} -1/3 & -1/3 & 1/3 \\ 4/3 & 7/3 & -1/3 \\ 1/6 & 1/6 & 1/3 \end{bmatrix}$.

27. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$E_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.