Solution to Some Practice Problems for Midterm Exam 2 (Math 18, Coo, Spring 17)

#2. $T: \mathbb{R}^4 \to \mathbb{R}^4$ linear, one-to-one. $T(\vec{x}) = A\vec{x}$.

Is A invertible? Yes. $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$. $Nul(A) = \{0\}$.

#3 LU-factorization

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \xrightarrow[(-3)(1)+(3)\to(3)]{(-1)(1)+(2)\to(2)} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \xrightarrow{5(2)+(3)\to(3)} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

Check: $LU = A$. $\vec{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$ $A\vec{x} = \vec{b}$

$LU\vec{x} = \vec{b}$. $U\vec{x} = \vec{y}$. $L\vec{y} = \vec{b}$ $\begin{cases} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{cases}$

$L\vec{y} = \vec{b}$
$$\begin{aligned} y_1 &= 0 \\ y_1 + y_2 &= -5 \Rightarrow y_2 = -5 \\ 3y_1 - 5y_2 + y_3 &= 7 \Rightarrow y_3 = 7 - 25 = -18 \end{aligned}$$

$U\vec{x} = \vec{y}$
$$\begin{aligned} -6x_3 &= -18, \quad x_3 = \boxed{3} \\ -2x_2 - x_3 &= -5 \Rightarrow x_2 = (-5 + x_3)/(-2) = \boxed{1} \\ x_1 - x_2 + 2x_3 &= 0, \quad x_1 = x_2 - 2x_3 = \cancel{-\tfrac{1}{2} - \tfrac{1}{3} = -\tfrac{5}{6}} \\ x_1 &= \boxed{-5} \end{aligned}$$

$\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$. Check: $A\vec{x} = \vec{b}$. Yes!

#4. Co-factors $C_{ij} = (-1)^{i+j} \det A_{ij}$. $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & -4 & 1 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 \end{bmatrix}$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 2 & -4 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} = 0$$

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 0 & -4 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix} = 0. \quad C_{23} = (-1)^{2+3} \det \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = 0$$

$$C_{44} = (-1)^{4+4} \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -4 \\ -1 & 0 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -4 \\ 0 & 1 & 4 \end{bmatrix} = 12.$$

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#5. (a) A, B equivalent: $A \sim \cdots \sim B$. $A = CBD$

C, D: invertible. Elementary row reductions can change the determinant.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = B \qquad \det A = -\det B = 1.$$

(b) No. $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$.

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\underset{1}{\det(A+B)} \neq \underset{0}{\det A} + \underset{0}{\det B}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $-A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det A = \det(-A) = 1$

But $A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $-A = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$ $\det A = 1$. $\det(-A) = -1$

(d) $\det A^T = \det A$ yes! $\det A^{-1} = \dfrac{1}{\det A}$ yes

$\boxed{\text{pf } A^{-1}A = I \quad \det A^{-1} \det A = \det I = 1.}$

(e) True: A is invertible $\Leftrightarrow \det A \neq 0$

#6.

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5 \begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} + (-1)^{1+3}(-1) \begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} = 5(-9+10) - 5 = \boxed{0}$$

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 0 \\ -1 & 3 & 12 & 0 \\ 1 & 2 & -4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 0 & -4 & 0 \\ 0 & 4 & 14 & -1 \\ 0 & 1 & -6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 0 \\ 4 & 14 & -1 \\ 1 & -6 & 2 \end{vmatrix} = 4 \begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} = 4(9) = \boxed{36}$$

$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & 5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -2 & 5 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 10 \\ -3 & -7 & -5 & 2 \end{vmatrix}$$

$$= (-1)^{3+4} 10 \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ -3 & -7 & -5 \end{vmatrix} = -10 \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 0$$

#7 $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a).$

#8. Cramer's rule (not a crazy rule): $A\vec{x} = \vec{b}$

$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots \quad x_n = \frac{\det A_n}{\det A}.$

What is A_2? You replace the 2nd col. of A by $\vec{b}$.

Now, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\det A = 2$. $C_{11} = 2$, $C_{12} = -4$, $C_{13} = 6$.

Find sol'n to $A\vec{x} = \vec{e}_1$ $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = \frac{\det A_1}{\det A}$ $A_1 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$ $\det A_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = 2$ So, $\det A_1 = 2$.

$x_1 = \frac{2}{2} = \boxed{1}$. $x_2 = \frac{\det A_2}{\det A}$. $A_2 = \begin{bmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$

$\det A_2 = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ $C_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -4$

$\Rightarrow \det A_2 = -4$ $x_2 = -4/2 = \boxed{-2}$

$x_3 = \frac{\det A_3}{\det A}$ $A_3 = \begin{bmatrix} * & * & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$. $\det A_3 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$C_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 6$

$\det A_3 = 6$ $x_3 = \frac{6}{2} = \boxed{3}$

#9. $A^{-1} = \frac{1}{\det A}\operatorname{adj} A \Rightarrow \underbrace{AA^{-1}}^{=I} = \frac{1}{\det A} A(\operatorname{adj} A) \Rightarrow \boxed{A(\operatorname{adj} A) = (\det A) I}$

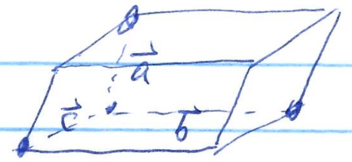
Correct.

#10 $\vec{a} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ $\vec{a} \times \vec{b}$

$\vec{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$. Area $= \left|\det\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}\right|$

$= |6 \cdot 3 - 1(-3)| = \boxed{15}$

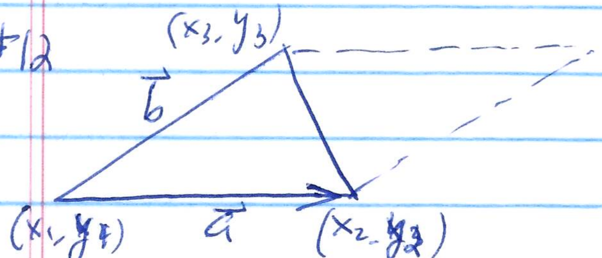
#11. $\vec{a} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}$



$$\left| \vec{a}\ \vec{b}\ \vec{c} \right| = \begin{vmatrix} 1 & -1 & -2 \\ 4 & 2 & -5 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -2 \\ 0 & 10 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 20 + 3 = \boxed{23} \qquad \text{Vol.} = 23.$$

#12



$$\vec{a} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{bmatrix}$$

Area of $\triangle$ = $\frac{1}{2}$ area of parallelogram $= \frac{1}{2} \left| \det [\vec{a}\ \vec{b}] \right|$

$= \frac{1}{2} \cdot$ abs. value of $\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$.

Now $\frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$= \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

$= \frac{1}{2} \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$. Thus, the abs. of this = area of the $\triangle$.

#13 (a) $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}, abc = 0 \right\}$. Not a subspace.

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in H$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in H$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin H$.

(b) $H = \left\{ \begin{bmatrix} 1 \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\}$. No. $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin H$.

(c) $H = \left\{ \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\}$ Yes. $\begin{bmatrix} 0 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \in H$

$a \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ ab \\ ac \end{bmatrix} \in H$.

(d) $H = \left\{ \begin{bmatrix} 2a - b \\ 2b - c \\ 2c - a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ Yes. $H = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$

In fact, this $H = \mathbb{R}^3$.

#14. $\{2\times 2 \text{ diag. matrices}\}$ is a subspace of $M^{2\times 2}$.

#15. $x_1\begin{bmatrix}1\\1\\-2\\0\end{bmatrix} + x_2\begin{bmatrix}-2\\0\\2\\2\end{bmatrix} + x_3\begin{bmatrix}1\\4\\-4\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\-2\end{bmatrix}$

$$\begin{bmatrix}1&-2&1&0\\1&0&4&1\\-2&2&-4&1\\0&2&1&-2\end{bmatrix} \sim \begin{bmatrix}1&-2&1&0\\0&2&3&1\\0&-2&-2&1\\0&2&1&-2\end{bmatrix} \sim \begin{bmatrix}1&-2&1&0\\0&2&3&1\\0&0&1&2\\0&0&-2&-3\end{bmatrix}$$

$\sim \begin{bmatrix}1&-2&1&0\\0&2&3&1\\0&0&1&2\\0&0&0&1\end{bmatrix}$ No sol'n. So, not in the span.

#16. $\begin{bmatrix}-2&4\\-1&2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ So, in the null space

$\begin{bmatrix}2\\1\end{bmatrix} = x_1\begin{bmatrix}-2\\-1\end{bmatrix} + x_2\begin{bmatrix}4\\2\end{bmatrix}$ $\begin{bmatrix}-2&4&2\\-1&2&1\end{bmatrix} \to \begin{bmatrix}1&-2&-1\\0&0&0\end{bmatrix}$

Yes.

#17. $\vec{v}_1, \vec{v}_2, a\vec{v}_1 + b\vec{v}_2$ are LD since:

Case 1: $a=b=0$ then $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$,

$0\vec{v}_1 + 0\vec{v}_2 + 1\cdot\vec{0} = \vec{0} \Longrightarrow$ LD

Say, $a\neq 0$

Case 2 a, b: at least one of them $\neq 0$.

So, $(-a)\vec{v}_1 + (-b)\vec{v}_2 + 1\cdot(a\vec{v}_1 + b\vec{v}_2) = \vec{0}$

$\neq 0$

Simply: $(-a)\vec{v}_1 + (-b)\vec{v}_2 + 1\cdot(a\vec{v}_1 + b\vec{v}_2) = \vec{0}$.

$1\neq 0$.

#18. $a\,p_0(t) + b\,p_1(t) + c\,p_2(t) = 0$ (zero polynomial)

$\Longrightarrow a + b(1+t) + c(1+t+t^2) = 0 \Longrightarrow (a+b+c) + (b+c)t + ct^2 = 0$

$\Longrightarrow \begin{cases} a+b+c=0 \\ b+c=0 \\ c=0 \end{cases} \Longrightarrow$ all $a=b=c=0 \Longrightarrow p_0, p_1, p_2$ are LI.

But $\dim \mathbb{P}_2 = 3$. So, $\{p_0, p_1, p_2\}$ is a basis.

Let $p(t) = 2 - t + 3t^2 = c_0 p_0(t) + c_1 p_1(t) + c_2 p_2(t)$

$2 - t + 3t^2 = c_0 + c_1(1+t) + c_2(1+t+t^2)$

$= (c_0 + c_1 + c_2) + (c_1 + c_2)t + c_2 t^2$

$$\Longrightarrow \begin{cases} c_0 + c_1 + c_2 = 2 \\ c_1 + c_2 = -1 \\ c_2 = 3 \end{cases} \Longrightarrow \begin{array}{l} c_2 = 3 \\ c_1 = -4 \\ c_0 = 3. \end{array}$$

The coordinates are $\begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}$.

#19 $A = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

A basis for Col(A): $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix} \right\}$ $\left(\text{Not} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$

rank(A) = 2. = dim Col(A)

~~A basis~~ $A\vec{x} = \vec{0} \Longrightarrow \vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So, no basis for Nul(A)

dim Nul(A) = 0.

$A^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \end{bmatrix}$

basis for row space of A: (1, -2), (-2, 7)

$B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 & 5 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{4} & -11 \end{bmatrix}$

a basis for Col(B) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

rank(B) = 3.

$B \sim \begin{bmatrix} 1 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11/4 \end{bmatrix}$

A basis for Nul(B) is $\begin{bmatrix} -9/4 \\ -2 \\ 11/4 \\ 1 \end{bmatrix}$.

rank(B) = dim Col(B) = 3.

$B^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

a basis for row space of B is (1, 0, -1, 5), (0, 1, 0, 2) and (2, 0, 2, -1)

#20. (a) Same null space. May be different col. spaces same row space.

(b) Yes, except we write row vectors. €

(c) Yes.

(d) Yes. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\vec{u}\vec{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1\, v_2\, v_3] = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

Any two rows are parallel. (Same for columns)

#21 $\dim \mathbb{R}^4 = 4$. $\dim \mathbb{P}_4 = 5$. $\dim M^{4\times 4} = 16$.

#22. Yes. Let $\dim V = n$ $\{\vec{v}_1, \dots \vec{v}_n\}$ a basis for V.
Any $n+1$ vectors $\vec{u}_1 \cdots \vec{u}_{n+1}$ in V must be LI.
To see the idea, consider $n=2$. $\vec{v}_1, \vec{v}_2$ a basis.
So. $\vec{u}_1 = a_1\vec{v}_1 + a_2\vec{v}_2$ $\vec{u}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$. $\vec{u}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$
Try $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \vec{0}$
So, $x_1(a_1\vec{v}_1 + a_2\vec{v}_2) + x_2(b_1\vec{v}_1 + b_2\vec{v}_2) + x_3(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}$
$\Rightarrow (a_1x_1 + b_1x_2 + c_1x_3)\vec{v}_1 + (a_2x_1 + b_2x_2 + c_2x_3)\vec{v}_2 = \vec{0}$
But, $\vec{v}_1, \vec{v}_2$ are LI. So. $a_1x_1 + b_1x_2 + c_1x_3 = 0$
$a_2x_1 + b_2x_2 + c_2x_3 = 0$
These equations must have non zero soln's x_1, x_2, x_3. So, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are LD.

#23 (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = B$ $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ rank A = rank B = 1
rank $(A+B) = 1$.
In general rank$(A+B) \neq$ rank A + rank B)

(b) In general rank$(AB) \neq$ rank(A) rank(B).
$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Yes. $\dim \mathrm{Col}(A) = $ # pivot variables
$\dim \mathrm{Nul}(A) = $ # free variables
pivot variables + # free variables = # variables
= # col. of A

24. Yes $A\vec{u} = \lambda\vec{u}$ $\quad \vec{u} \neq \vec{0}$.

$$A^2\vec{u} = A(\lambda\vec{u}) = \lambda A\vec{u} = \lambda^2 \cancel{A}\vec{u}. \quad (\vec{u} \neq \vec{0})$$

25. Yes. $A\vec{u}_1 = \lambda_1\vec{u}_1$. $A\vec{u}_2 = \lambda_2\vec{u}_2$ $\quad A\vec{u}_3 = \lambda_3\vec{u}_3$.

all $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are nonzero vectors.

$\lambda_1 \neq \lambda_2,\ \lambda_2 \neq \lambda_3,\ \lambda_3 \neq \lambda_1$

Let's say $\quad c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$

Then $\quad A(\text{--------}) = A\vec{0} = \vec{0}$

$$c_1A\vec{u}_1 + c_2A\vec{u}_2 + c_3A\vec{u}_3 = \vec{0}$$

$$c_1\lambda_1\vec{u}_1 + c_2\lambda_2\vec{u}_2 + c_3\lambda_3\vec{u}_3 = \vec{0}$$

On the other hand

$$c_1\lambda_1\vec{u}_1 + c_2\lambda_1\vec{u}_2 + c_3\lambda_1\vec{u}_3 = \vec{0}$$

Subtracting one eq from the other, we get

$$c_2(\lambda_1-\lambda_2)\vec{u}_2 + c_3(\lambda_1-\lambda_3)\vec{u}_3 = \vec{0}.$$

Now $A\ (\text{——————}) = A\vec{0} = \vec{0}$

$$c_2(\lambda_1-\lambda_2)\lambda_2\vec{u}_2 + c_3(\lambda_1-\lambda_3)\lambda_3\vec{u}_3 = \vec{0}$$

$\lambda_2:\quad c_2(\lambda_1-\lambda_2)\lambda_2\vec{u}_2 + c_3(\lambda_1-\lambda_3)\lambda_2\vec{u}_3 = \vec{0}$

Subtract to get $\quad c_3\underbrace{(\lambda_1-\lambda_3)}_{\neq 0}\underbrace{(\lambda_2-\lambda_3)}_{\neq 0}\underbrace{\vec{u}_3}_{\neq\vec{0}} = \vec{0} \Rightarrow c_3 = 0$

~~So~~ Hence $\quad c_2(\lambda_1-\lambda_2)\vec{u}_2 = \vec{0} \Rightarrow c_2 = 0.$

$\Rightarrow c_1\underbrace{\vec{u}_1}_{\neq\vec{0}} = \vec{0} \Rightarrow c_1 = 0.$ All $c_1 = c_2 = c_3 = 0.$

So $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are LI.

#26. $A = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$ $\quad \det(A-\lambda I) = \begin{vmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = \cancel{}(9-\lambda)(5-\lambda)+4$

$= \lambda^2 - 14\lambda + \cancel{49} \; 49 = (\lambda-7)^2 = 0$ $\quad \boxed{\lambda_1 = \lambda_2 = 7}$

$(A-\lambda_1 I) = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, only $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ one LI e-vector.

#27. $\det(A-\lambda I) = (3-\lambda)(\lambda^2-15\lambda+50) = (3-\lambda)(\lambda-5)(\lambda-10)$

$\lambda_1 = 3,\ \lambda_2 = 5,\ \lambda_3 = 10.$

$\lambda_1 = 3 \quad A - \lambda_1 I = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 6 & 0 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 5 \quad A - \lambda_2 I = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 5 & 8 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 18 & -2 \end{bmatrix}$

$x_1 - 2x_2 = 0. \quad 18x_2 - 2x_3 = 0 \qquad x_3 = 9x_2. \quad x_1 = 2x_2$

$\vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}.$

$\lambda_3 = 10 \quad A - \lambda_3 I = \begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \\ 5 & 8 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 5 & 8 & -7 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 10 & 16 & -14 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 11 & -14 \\ 0 & 0 & 0 \end{bmatrix}$

$2x_1 + x_2 = 0 \qquad x_2 = 14, \quad x_1 = -7. \quad x_3 = \frac{1}{14} \cdot 11x_2 = 11.$

$11x_2 - 14x_3 = 0. \qquad \vec{u}_3 = \begin{bmatrix} -7 \\ 14 \\ 11 \end{bmatrix}.$