MATH 20D, Section C00, Spring Quarter, 2005
MIDTERM EXAM 2

Solutions and Grading Guidelines

1. (30 points) Consider the differential equation

$$dy/dt = y\left(1-y^2\right), \qquad 0 \le t < \infty.$$

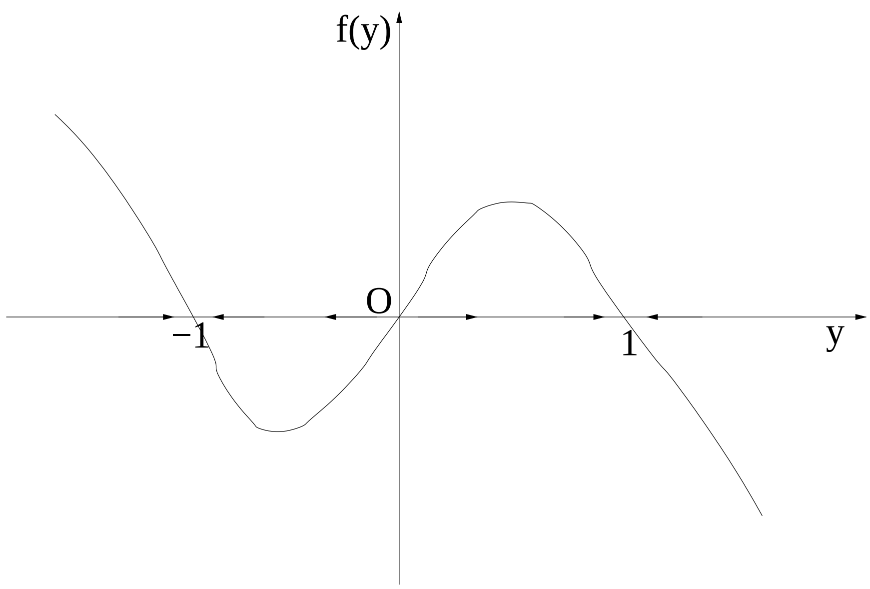
   (1) Sketch the graph of $f(y) = y\left(1-y^2\right)$ versus y.
   (2) Find all the stationary points of the differential equation, and classify each of them as asymptotically stable or unstable as $t \to +\infty$.
   (3) On the same (t, y)-plane, sketch the solutions with the initial values

$$y(0) = -1.5,\ -1,\ -0.5,\ 0,\ 0.5,\ 1,\ 1.5.$$

   Indicate clearly whether a solution is increasing, or decreasing, or constant, or none of them.

Solution.

(1) See the figure below.



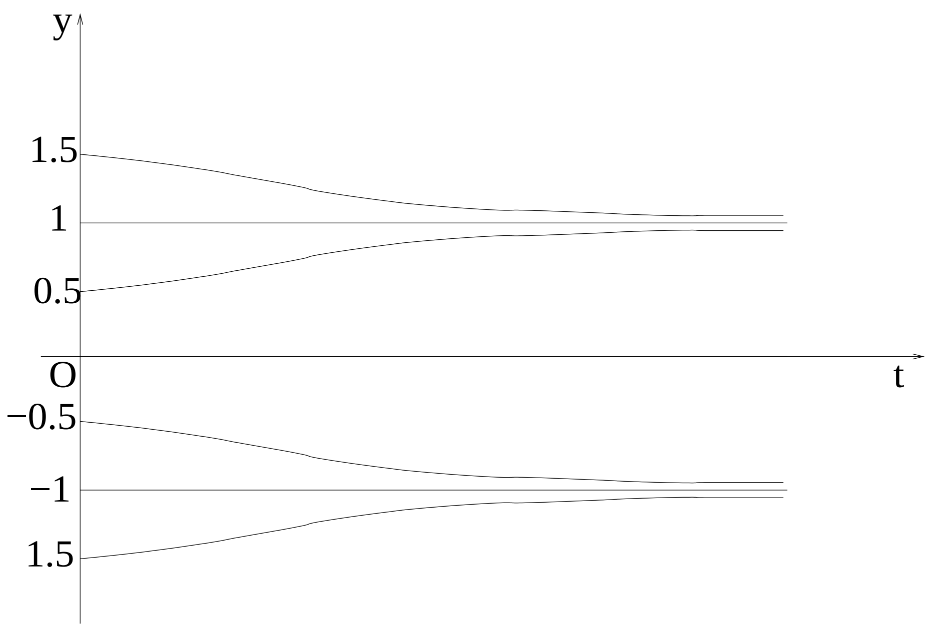
(2)
$y = -1$: asymptotically stable.
$y = 0$: asymptotically unstable.
$y = 1$: asymptotically stable.

(3) See the figure below.

2. (20 points) Consider the system of equations

$$\begin{cases} x' = x(1-y), \\ y' = y(1-x). \end{cases}$$

   (1) Find all its x-nullclines, y-nullclines, and stationary points.
   (2) Find an integral of the system.



Solution.

(1)
x-nullclines: $x = 0$ and $y = 1$.
y-nullclines: $y = 0$ and $x = 1$.
stationary points: $(0, 0)$, $(1, 1)$.

(2)

$$\frac{dy}{dx} = \frac{y(1-x)}{x(1-y)}.$$

This is separable.

$$\frac{1-x}{x}dx = \frac{1-y}{y}dy.$$

So,

$$\left(\frac{1}{x} - 1\right) dx = \left(\frac{1}{y} - 1\right) dy.$$

Integrate to get

$$\ln|x| - x = \ln|y| - y + c_1.$$

Thus,

$$\ln\left|\frac{y}{x}\right| = y - x + c_1.$$

Therefore,

$$\frac{ye^x}{xe^y} = C.$$

Final solution:

$$F(x, y) = \frac{ye^x}{xe^y},$$

or

$$F(x, y) = \frac{xe^y}{ye^x}.$$

3. (20 points) Find the general solution of the system of equations

$$\begin{cases} x' = 5x - 2y, \\ y' = 2x + y. \end{cases}$$

Solution.

$$A = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}$$

Eigenvalues:

$$\det(A - rI) = det \begin{bmatrix} 5-r & -2 \\ 2 & 1-r \end{bmatrix} = (r-3)^2 = 0.$$

We thus have the repeated eigenvalue: $r = 3$.

Let $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have

$$A\mathbf{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \neq 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = r\mathbf{c}.$$

Let

$$\mathbf{b} = A\mathbf{c} - r\mathbf{c} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The general solution is

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= C_1 e^{3t}\mathbf{b} + C_2 e^{3t}\left(t\mathbf{b} + \mathbf{c}\right) \\ &= e^{3t}\left(C_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 2t+1 \\ 2t \end{bmatrix} \right) \\ &= \begin{bmatrix} (2C_1 + C_2(2t+1))e^{3t} \\ (2C_1 + 2C_2 t)e^{3t} \end{bmatrix}. \end{aligned}$$

4. (20 points) Let A be a 2×2 matrix with real entries. Suppose it has the following two characteristic values and the corresponding characteristic vectors:

$$r_1 = -1 + i, \quad \mathbf{b}_1 = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \qquad \text{and} \qquad r_2 = -1 - i, \quad \mathbf{b}_2 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}.$$

(1) Find the real-valued general solution of the system of equations

$$\mathbf{v}' = A\mathbf{v}, \qquad \text{where } \mathbf{v} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

(2) Find the solution of this system of equations that satisfies the initial condition

$$\mathbf{v}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution.

(1) Since

$$\begin{aligned}
&e^{-t}(\cos t + i\sin t)\left(\begin{bmatrix}2\\1\end{bmatrix} + i\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= e^{-t}\left(\cos t\begin{bmatrix}2\\1\end{bmatrix} - \sin t\begin{bmatrix}1\\0\end{bmatrix}\right) + ie^{-t}\left(\sin t\begin{bmatrix}2\\1\end{bmatrix} + \cos t\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= e^{-t}\begin{bmatrix}2\cos t - \sin t\\ \cos t\end{bmatrix} + ie^{-t}\begin{bmatrix}2\sin t + \cos t\\ \sin t\end{bmatrix},
\end{aligned}$$

the real-valued general solution is

$$\mathbf{v}(t) = C_1 e^{-t}\begin{bmatrix}2\cos t - \sin t\\ \cos t\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}2\sin t + \cos t\\ \sin t\end{bmatrix}.$$

(2)

$$C_1\begin{bmatrix}2\\1\end{bmatrix} + C_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Thus,

$$\begin{aligned}
2C_1 + C_2 &= 0,\\
C_1 &= 1.
\end{aligned}$$

Thus, $C_1 = 1$ and $C_2 = -2$.

The final solution is:

$$\mathbf{v}(t) = e^{-t}\begin{bmatrix}2\cos t - \sin t\\ \cos t\end{bmatrix} - 2e^{-t}\begin{bmatrix}2\sin t + \cos t\\ \sin t\end{bmatrix} = \begin{bmatrix}-5e^{-t}\sin t\\ e^{-t}(\cos t - 2\sin t)\end{bmatrix}.$$

5. (10 points) Consider the differential equation

$$(\cos t)y'' - 4ty' + y = 2e^t.$$

(1) Find all the singular points of the equation.
(2) Find the domain of the initial-value problem of this differential equation with the initial conditions $y(1) = 0$ and $y'(1) = 1$.

Solution.

(1) Let $\cos t = 0$. So, all the singular points are:

$$t = \pi/2 + n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots .$$

(2) Since, $t_0 = 1$ and $-\pi/2 < 1 < \pi/2$, the domain is $(-\pi/2, \pi/2)$.