Math 210A, Fall 2009. Homework Assignment 1. Due Friday, October 9, 2009

1. Let $\mathbb{R}^\infty$ denote the set of all the real sequences $\{\xi_k\}_{k=1}^\infty$. Two elements $\{\xi_k\}_{k=1}^\infty, \{\eta_k\}_{k=1}^\infty \in \mathbb{R}^\infty$ are equal if $\xi_k = \eta_k$ for all $k = 1, 2, \ldots$ Define the addition and scalar multiplication by $\{\xi_k\}_{k=1}^\infty + \{\eta_k\}_{k=1}^\infty = \{\xi_k + \eta_k\}_{k=1}^\infty$ and $\alpha\{\xi_k\}_{k=1}^\infty = \{\alpha\xi_k\}_{k=1}^\infty$ for all $\{\xi_k\}_{k=1}^\infty, \{\eta_k\}_{k=1}^\infty \in \mathbb{R}^\infty$ and all $\alpha \in \mathbb{R}$.
   (1) Prove that $\mathbb{R}^\infty$ is an infinitely dimensional real vector space.
   (2) Define $l_2 = \left\{\{\xi_k\}_{k=1}^\infty \in \mathbb{R}^\infty : \sum_{k=1}^\infty \xi_k^2 < \infty\right\}$. Prove that l_2 is an infinitely dimensional subspace of $\mathbb{R}^\infty$.

2. Let $n \geq 2$ be an integer. Let $r_1, \ldots, r_n$ be n real numbers. Define the functions $f_k : [0,1] \to \mathbb{R}$ by $f_k(x) = e^{r_k x}$ $(k = 1, \ldots, n)$. Prove that $f_1, \ldots, f_n$ are linearly independent in the real vector space $C[0,1]$ which consists of all real, continuous functions on $[0,1]$.

3. Find a basis of the subspace that is complementary to the subspace Span $\{u_1, u_2\}$ in the real vector space $\mathbb{R}^4$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad u_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

4. Prove that any finitely many, non-zero, mutually orthogonal vectors in an inner-product space are linearly independent.

5. Consider the inner-product space $L^2(-1,1)$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx \qquad \forall f, g \in L^2(0,1).$$

   Let $f_0(x) = 1$, $f_1(x) = 1 + x$, and $f_2(x) = 1 + x + x^2$.
   (1) Show that f_0, f_1, f_2 are linearly independent.
   (2) Use the Gram–Schmidt orthogonalization process to construct an orthonormal basis for the subspace $E_2 = \text{Span}\,\{f_0, f_1, f_2\}$.