Math 210A, Fall 2009. Homework Assignment 2. Due Friday, October 23, 2009

1. Let $p_0, p_1, p_2 \in \mathcal{P}_2$ be given by

$$p_0(x) = 1, \qquad p_1(x) = 1 + x, \qquad p_2(x) = 1 + x + x^2.$$

   (1) Show that $\{p_0, p_1, p_2\}$ is a basis for $\mathcal{P}_2$.
   (2) Find the corresponding dual basis $\{p_0^*, p_1^*, p_2^*\}$ for $\mathcal{P}_2^*$.

2. Let X be an inner-product space over $\mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $T \in \mathcal{L}(X)$.

   (1) Prove that $T = 0$ if and only if $\langle u, Tv\rangle = 0$ for all $u, v \in X$.
   (2) Assume that $\mathbb{F} = \mathbb{C}$. Prove that $T = 0$ if and only if $\langle u, Tu\rangle = 0$ for all $u \in X$.
   (3) Assume that $\mathbb{F} = \mathbb{R}$. Assume also that $\langle Tu, u\rangle = 0$ for all $u \in X$. Is it still true that $T = 0$?

3. (Heisenberg's Uncertainty Principle.) Let X be an inner-product space and $u \in X$. Let $T, S \in \mathcal{L}(X)$ be self-adjoint. Denote $\langle S\rangle_u = \langle u, Su\rangle$ and $\langle T\rangle_u = \langle u, Tu\rangle$. Denote

$$(\Delta S)_u = \sqrt{\langle u, (S - \langle S\rangle_u I)^2 u\rangle} \qquad \text{and} \qquad (\Delta T)_u = \sqrt{\langle u, (T - \langle T\rangle_u I)^2 u\rangle},$$

   where $I \in \mathcal{L}(X)$ is the identity operator. Prove that

$$(\Delta S)_u (\Delta T)_u \geq \frac{1}{2} \left|\langle u, [S, T]u\rangle\right|.$$

4. Prove the following properties of the tensor product on a real inner-product space X:

   (1) For any $u, v \in X$ and $T \in \mathcal{L}(X)$, $T(u \otimes v) = (Tu) \otimes v$ and $(u \otimes v)T = u \otimes (T^* v)$;
   (2) For any $u, v, w, z \in X$, $(u \otimes v)(w \otimes z) = \langle v, w\rangle (u \otimes z)$.

5. Find the matrix e^A with $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.