Math 210A, Fall 2009. Homework Assignment 3. Due Friday, November 6, 2009

1. Let X be an inner-product space. Let $u_k \to u$ and $v_k \to v$ in X. Prove that $\|u_k\| \to \|u\|$ and that $\langle u_k, v_k \rangle \to \langle u, v \rangle$.

2. In a finitely-dimensional Hilbert space, any bounded sequence has a convergent subsequence. Is this still true in an infinite-dimensional Hilbert space?

3. Define for each integer $n \geq 0$ Chebyshev polynomials are defined by

$$T_n(x) = \cos(n \arccos x) \qquad \forall x \in [-1, 1], \qquad n = 0, 1, \ldots$$

   (1) Prove that, for all $x \in [-1, 1]$, $T_0(x) = 1$, $T_1(x) = x$, and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 2, 3, \ldots$$

   (2) Write down explicitly $T_2(x)$ and $T_3(x)$.

4. The first few Lengendre polynomials are given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

   Use these polynomials to find the least-squares approximation of the function $f(x) = x^4$ by polynomials of degrees ≤ 3 on $(-1, 1)$, i.e., find the polynomial $p \in \mathcal{P}_3$ such that

$$\int_{-1}^{1} |x^4 - p(x)|^2 dx \leq \int_{-1}^{1} |x^4 - q(x)|^2 dx \qquad \forall q \in \mathcal{P}_3.$$

5. Let H be a Hilbert space with a complete orthonormal system of vectors $\{u_k\}_{k=1}^{\infty}$. Define $T : l_2 \to H$ by $T\xi = \sum_{k=1}^{\infty} \xi_k u_k$ for all $\xi = (\xi_1, \xi_2, \ldots) \in l_2$. Show that T is an isomorphism between l_2 and H, i.e., $T : l_2 \to H$ is one-to-one, onto, linear, and inner-product preserving (i.e., $\langle T\xi, T\eta \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in l_2$).