Math 210A, Fall 2009. Homework Assignment 4. Due Friday, November 20, 2009

1. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in C[a,b]$. Assume that
$$\int_a^b x^k f(x)\,dx = 0 \qquad \text{for all } k = 0, 1, \dots$$
Show that $f(x) = 0$ for all $x \in [a,b]$.

2. Verify that the Chebyshev polynomial $T_n(x)$ satisfies the differential equation
$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0.$$

3. Prove the following properties of the Chebyshev Polynomials of second kind
$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n = 0, 1, \dots,$$
where $x = \cos\theta$ ($\theta \in [0,\pi]$):

   (1) *Recursion formula.*
$$\begin{aligned} &U_0(x) = 1, \quad U_1(x) = 2x, \\ &U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n = 1, 2, \dots; \end{aligned}$$

   (2) *Orthogonality.*
$$\int_{-1}^{1} \sqrt{1-x^2}\,U_m(x)U_n(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi/2 & \text{if } m = n; \end{cases}$$

   (3) *Differential equations.*
$$\left(1-x^2\right)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0 \qquad n = 0, 1, \dots;$$

   (4) *Relations with the Chebyshev polynomials of first kind.*
$$\begin{aligned} &nU_{n-1}(x) = T_n'(x), \qquad n = 1, 2, \dots, \\ &U_n(x) = xU_{n-1}(x) + T_n(x), \qquad n = 1, 2, \dots; \end{aligned}$$

   (5) For each $n \geq 0$, U_n is a polynomial of degree n with leading coefficient 2^n. Moreover, if n is even (odd), then U_n is an even (odd) polynomial.

4. Let $\mathbf{r}$ be a non-zero vector along the z-axis. Let $\mathbf{r}'$ be any vector. Use the generating function for Legendre polynomials $P_n(x)$ ($n = 0, 1, \dots$) to show that
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\max(|\mathbf{r}|, |\mathbf{r}'|)} \sum_{n=0}^{\infty} \left(\frac{\min(|\mathbf{r}|, |\mathbf{r}'|)}{\max(|\mathbf{r}|, |\mathbf{r}'|)} \right)^n P_n(\cos\theta),$$
where θ is the polar angle of $\mathbf{r}'$.

5. Find the Fourier series expansion of the period function f defined by $f(\theta) = -(\pi+\theta)/2$ if $-\pi \leq \theta < 0$ and $f(\theta) = (\pi-\theta)/2$ if $0 < \theta \leq \pi$.