

Math 210A, Fall 2009. Homework Assignment 5. Due Friday, December 4, 2009

1. Prove the following identity:

$$\frac{1}{2} + \cos \theta + \cdots + \cos n\theta = \frac{\sin((n+1/2)\theta)}{2 \sin \theta/2} \quad \forall \theta \in \mathbb{R}, \forall n = 1, 2, \dots$$

(Remark: for $\theta = 0$ the right-hand side of the identity is interpreted as the limiting value as $\theta \rightarrow 0$.)

2. Let $f(\theta)$ be a 2π -periodic function such that $f(\theta) = \theta^2$ for $\theta \in [-\pi, \pi]$.
- (1) Find the Fourier series of f .
 - (2) Show by Dini's test that f and its Fourier series are equal to each other at every point in \mathbb{R} .
 - (3) By evaluating both f and its Fourier series at some points to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

3. Suppose $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ is the (complex) Fourier series of a Lipschitz continuous function $f(\theta)$. Find the Fourier transform of f .
4. Let $N \geq 1$ be an integer and $x_j = j2\pi/N$, $j = 0, \dots, N-1$. Show for any integers k and l that

$$\sum_{j=0}^{N-1} e^{ikx_j} e^{-ilx_j} = \begin{cases} N & \text{if } k \equiv l \pmod{N}, \\ 0 & \text{if } k \not\equiv l \pmod{N}. \end{cases}$$

(Remark: $k \equiv l \pmod{N}$ means that $k - l$ is an integer multiple of N .)

5. The displacement of a damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t} e^{i\omega_0 t} & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases}$$

where A , α , and ω_0 are all real numbers.

- (1) Find the Fourier transform $\tilde{f}(\omega)$ of f .
- (2) Show that the frequency distribution $|\tilde{f}(\omega)|^2$ is given by

$$|\tilde{f}(\omega)|^2 = \frac{A^2}{2\pi [(\omega - \omega_0)^2 + \alpha^2]}.$$