Math 210. Lecture Notes. Bo Li, Fall 2013

Chapter 1. Sequences and Series

Section 1.1 Sequences of numbers
Section 1.2 Infinite series of numbers
Section 1.3 Sequences of functions
Section 1.4 Infinite series of functions. Power series

Section 1.1 Sequences of Numbers

Notation. $\mathbb{R}$ = the set of all real numbers
$\mathbb{C}$ = the set of all complex numbers

Examples

(1) $a_n = \frac{1}{n}$ $(n = 1, 2, \cdots)$

(2) $b_n = n$ $(n = 1, 2, \cdots)$

(3) $c_n = \sqrt{n+1} - \sqrt{n}$ $(n = 1, 2, \cdots)$

(4) $d_n = \sin\left(\frac{n\pi}{2}\right)$ $(n = 1, 2, \cdots)$

(5) $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n = 2, 3, \cdots)$

(6) $u_n = e^{in\theta}$ $(0 < \theta < \frac{\pi}{2}$: fixed$)$ $(n = 1, 2, \cdots)$

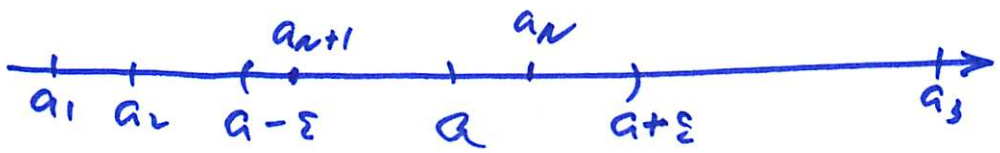
(7) $\pi_1 = 3.1$, $\pi_2 = 3.14$, $\pi_3 = 3.141$
$\pi_4 = 3.1415$, $\pi_5 = 3.14159$, $\pi_6 = 3.141592$
$\pi_7 = 3.1415926$, $\pi_8 = 3.14159265$
$\pi_9 = 3.141592653, \cdots$
$\pi_{14} = 3.14159265358979.$
$\cdots$

<u>Definition</u> (1) A sequence of numbers $\{a_n\}$ converges to a number a, denoted $\lim_{n\to\infty} a_n = a$, or $a_n \to a$ as $n \to \infty$, or simply $a_n \to a$, if for any $\varepsilon > 0$, there exists N (depending on ε) such that

$$n \geq N \Longrightarrow |a_n - a| < \varepsilon .$$

<u>Remark</u>

① $|a_n - a| < \varepsilon \Longleftrightarrow a - \varepsilon < a_n < a + \varepsilon$



② Smaller $\varepsilon > 0 \Longrightarrow$ large $N = N(\varepsilon)$.

(2) (Cauchy sequence) A sequence of numbers $\{a_n\}$ is a Cauchy sequence, if for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$n, m \geq N \Longrightarrow |a_n - a_m| < \varepsilon .$$

(3) A sequence of numbers $\{a_n\}$ is <u>bounded</u> if there exists $M > 0$ such that

$$|a_n| \leq M \quad (n = 1, 2, \cdots)$$

Or, exist m and M, two real numbers, such that

$$m \leq a_n \leq M \quad (n = 1, 2, \cdots)$$

a lower bound an upper bound

(4) A sequence of <u>real</u> numbers $\{a_n\}$ is monotonically increasing if

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots$$

strictly monotonically increasing if

$$a_1 < a_2 < \cdots < a_n < \cdots$$

Similarly define: monotonically decreasing
strictly decreasing

<u>monotonic</u> if either monotonically increasing or monotonically decreasing.

<u>Some basic facts</u>

(1) If $\lim_{n\to\infty} a_n$ exists (i.e., $\{a_n\}$ converges) then:
- the limit is unique. (Prove this!)
- $\{a_n\}$ is bounded. (Prove this!)
- any subseq $\{a_{n_k}\}$ converges to the same limit. (Prove this!)

(2) (A big) <u>Theorem</u> A seq. of numbers $\{a_n\}$ converges $\Longleftrightarrow$ $\{a_n\}$ is a Cauchy sequence.

<u>Proof</u> "$\Longrightarrow$" Let $\varepsilon > 0$. Since $a_n \to a$ for some a, there exists N such that

$$n \ge N \Longrightarrow |a_n - a| < \varepsilon/2.$$

Thus, if both $m, n \ge N$, we must

$|a_n - a| < \varepsilon/2$ and $|a_m - a| < \varepsilon/2$

Hence

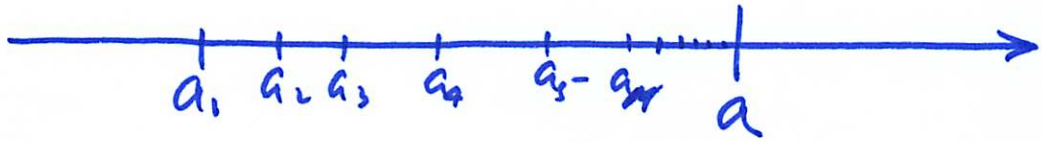
$$|a_m - a_n| = |(a_m - a) + (a - a_n)|$$

$$\leq |a_m - a| + |a - a_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \underline{\text{Q.E.D.}}$$

End of proof.

The direction "$\Leftarrow$" is more involved.

(3) <u>Theorem</u> Any monotonically increasing or decreasing, and bounded sequence (of real numbers) converges.



e.g., $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq a$

The limit is the smallest upper bound of $\{a_n\}$ (if $a_n \uparrow$).

Notation: $a_n \uparrow$ means $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$

$a_n \downarrow$ means $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$

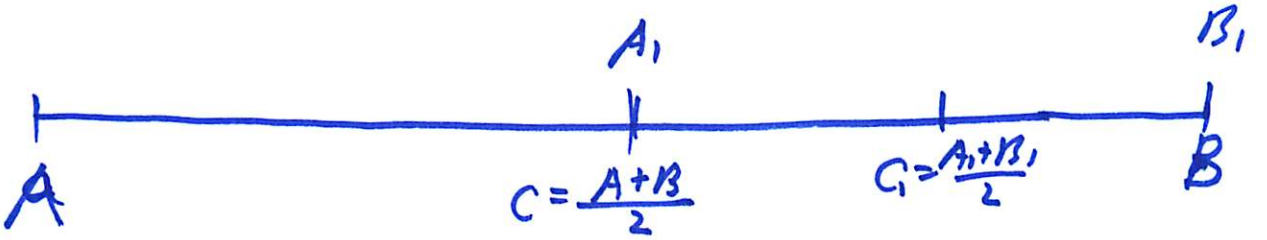
(4) <u>Theorem</u> Any bounded sequence of numbers $\{a_n\}$ must contain a convergent subsequence $\{a_{n_k}\}$.

<u>Example</u> $(-1)^n$: $-1, 1, -1, \cdots$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & & \\ -1 & -1 & -1 & \longrightarrow & -1. \end{array}$$

Why so ?

Let's say $A \le a_n \le B$ $(n = 1, 2, \cdots)$



At least one of $[A, C]$, $[C, B]$ contains infinitly many terms a_n. Pick up one of them, call it $[A_1, B_1]$. pick up $a_{n_1} \in [A_1, B_1]$.

Do the same for $[A_1, B_1]$. Pick up $[A_2, B_2]$ and $a_{n_2} \in [A_2, B_2]$ with $n_2 > n_1$.

So, $[A_1, B_1] \supseteq [A_2, B_2] \supseteq \cdots \supseteq [A_k, B_k] \supseteq \cdots$

a_{n_1} $\quad a_{n_2}$ $\quad a_{n_k}$

length of $[A_k, B_k] = \frac{1}{2^k}(B - A) \to 0$ as $k \to \infty$.

Thus, $\{a_{n_k}\}$ is a Cauchy sequence. Therefore it converges. <u>Q.E.D.</u>

(5) <u>The Sequeeze Theorem</u>: Suppose

(a) $a_n \le b_n \le c_n$ $(n = 1, 2, \cdots)$

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A$.

Then $\lim_{n\to\infty} b_n = A$.

Prove it using the definition of convergence of sequences.

Some useful limits

(1) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = \sum_{k=0}^{\infty}\frac{1}{k!}$

[(2) $\lim_{x\to 0}\frac{\sin x}{x} = 1$]

(3) $\lim_{n\to\infty}\sqrt[n]{n} = 1$

(4) For any $a>0$: $\lim_{n\to\infty}\sqrt[n]{a} = 1$.

(5) For any $p>0$: $\lim_{n\to\infty}\frac{1}{n^p} = 0$

(6) For any $p>0$, any $\alpha\in\mathbb{R}$: $\lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n} = 0$

(7) For any q, $|q|<1$: $\lim_{n\to\infty} q^n = 0$.

$\ln n \to \infty$: slowly

$n \to \infty$: fast

$2^n \to \infty$: much faster

<u>Questions</u> (1) $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, ... converges?

(2) $f(n) = \frac{n^2+1}{n^4+3n+1}$. $(n=1,2,\dots)$

How to prove $f(n)\downarrow$ for $n\geq n_0$ for some n_0?

— Use L'Hospitale's Rule.

(3) Suppose $a_n \to a$. Let

$S_n = \frac{1}{n}(a_1+a_2+\dots+a_n)$. Does $S_n \to a$?