

## Chapter 1. Sequences and Series

### Section 1.2 Infinite Series of Numbers

Example (Geometric series) Let  $|r| < 1$ . ( $r \neq 0$ )

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$1+r+\dots+r^n = \frac{1-r^{n+1}}{1-r}$$

Naturally define

$$1+r+\dots+r^N+\dots = \lim_{N \rightarrow \infty} (1+r+\dots+r^n)$$

$$= \lim_{N \rightarrow \infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}.$$

Question How about  $r+r^2+r^3+\dots+r^n = ?$   
 Call it  $S_n$ . Then  $(1-r)S_n = S_n - rS_n = r+r^2+\dots+r^n - r^{n+1} = \dots$ .

Definition Let  $a_1, a_2, a_3, \dots$  be (real or complex numbers). Consider  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$

For each integer  $n \geq 1$ , define

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n.$$

$$\text{So, } S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

...

Call  $S_n$  a partial sum.

Definition.  $\sum_{n=1}^{\infty} a_n$  converges if  $\{s_n\}$  converges.  
 (diverges) (diverges)

Theorem (Cauchy criterion)  $\sum_{n=1}^{\infty} a_n$  converges  
 $\Leftrightarrow$  for any  $\varepsilon > 0$ , there exists  $N$  such that  
 $n \geq N, p \geq 1 \Rightarrow \left| \sum_{k=n+1}^{n+p} a_k \right| = |a_{n+1} + \dots + a_{n+p}| < \varepsilon.$

This is  $|s_{n+p} - s_n| < \varepsilon$ .

Note that  $a_n = s_{n+1} - s_n$ . So we have

Theorem  $\sum_{n=1}^{\infty} a_n$  converges  $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ .

Examples (1) If  $|r| > 1$ , then  $\sum_{n=0}^{\infty} r^n$  diverges.  
 since  $\lim_{n \rightarrow \infty} r^n$  does not exist.

(2)  $\sum_{n=1}^{\infty} \frac{1}{n^n}$  diverges, since  $\frac{1}{n^n} \rightarrow 0$  as  $n \rightarrow \infty$ .

(3)  $\frac{1}{n} \rightarrow 0$  but still  $\sum \frac{1}{n}$  diverges.

Definition (1)  $\sum_{n=1}^{\infty} a_n$  converges absolutely if  
 $\sum_{n=1}^{\infty} |a_n|$  converges.

(2)  $\sum_{n=1}^{\infty} a_n$  converges conditionally if

$\sum_{n=1}^{\infty} a_n$  converges but  $\sum_{n=1}^{\infty} |a_n|$  diverges.

Examples  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$  converges absolutely  
 [This will be proved later.]

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$  converges conditionally.

Theorem  $\sum_{n=1}^{\infty} a_n$  converges absolutely  
 $\Rightarrow \sum_{n=1}^{\infty} |a_n|$  converges.

Proof Use the Cauchy criterion and  
 $\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \left| \sum_{k=n+1}^{n+p} |a_k| \right|. \quad \underline{\text{Q.E.D.}}$

So, positive series are important.

An interesting example

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots = \infty.$$

This is equivalent to  $\lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}) = \infty$ .

We show more:

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \right) = \gamma$$

exists, called Euler's constant.

Proof Let  $B_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$

$$B_{n+1} - B_n = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) = \int_n^{n+1} \left(\frac{1}{x+1} - \frac{1}{x}\right) dx < 0.$$

So.  $B_n \downarrow$ . (i.e.,  $\{B_n\}$  monotonically decreases.)

$$\begin{aligned} \text{For } n \geq 2, \quad B_n &= \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n} - \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx \\ &= \frac{1}{n} + \sum_{k=1}^{n-1} \int_k^n \left(\frac{1}{k} - \frac{1}{x}\right) dx > 0. \end{aligned}$$

So.  $\{B_n\}$  is bounded below (by 0). Thus,

$$\lim_{n \rightarrow \infty} B_n \text{ exists. } \underline{\text{Q.E.D.}}$$

Positive series (each term  $a_n$  is positive)

When a positive series converges?

Examples (1)  $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ .

(2)  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  (p: fixed)

(3)  $\sum_{n=1}^{\infty} \frac{2n^2 + 4n - 5}{7n^5 + 2n + 1}$

Theorem Suppose  $a_n > 0$  ( $n = 1, 2, \dots$ ). Then

$\sum_{n=1}^{\infty} a_n$  converges  $\Leftrightarrow \{s_n\}$  is bounded.

Recall:  $s_n = \sum_{k=1}^n a_k$ , ( $n = 1, 2, 3, \dots$ ).

Proof  $\sum_{n=1}^{\infty} a_n$  converges means  $\{s_n\}$  converges.

But  $a_n > 0$  ( $n = 1, 2, \dots$ ) implies that  $s_n \uparrow$ .

In this case,  $\{s_n\}$  converges  $\Leftrightarrow \{s_n\}$  is bounded (above). Q.E.D.

Theorem (Comparison Test) Suppose  
 $0 \leq a_n \leq b_n$  ( $n = 1, 2, 3, \dots$ )

Then

(1)  $\sum_{n=1}^{\infty} b_n$  converges  $\Rightarrow \sum_{n=1}^{\infty} a_n$  converges.

(2)  $\sum_{n=1}^{\infty} a_n$  diverges  $\Rightarrow \sum_{n=1}^{\infty} b_n$  diverges.

Pf Consequence of previous Theorem. Q.E.D.

Theorem (Comparison Test — limit version).

Assume  $a_n > 0, b_n > 0$  ( $n = 1, 2, 3, \dots$ )

and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$ .

(1)  $0 < l < \infty$ :  $\sum_{n=1}^{\infty} a_n$  converges  $\Leftrightarrow \sum_{n=1}^{\infty} b_n$  converges.

(2)  $l = 0$ :  $\sum b_n$  converges  $\Rightarrow \sum a_n$  converges  
 $\sum a_n$  diverges  $\Rightarrow \sum b_n$  diverges

(3)  $l = \infty$ :  $\sum a_n$  converges  $\Rightarrow \sum b_n$  converges  
 $\sum b_n$  diverges  $\Rightarrow \sum a_n$  diverges.

Proof (1) For  $\varepsilon = l/2$ , there exists  $N$  such that  $n \geq N \Rightarrow |\frac{a_n}{b_n} - l| < l/2$ . i.e.

$$\frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \quad \text{for all } n \geq N.$$

$$\text{i.e., } a_n \geq \frac{l}{2} b_n > 0, b_n \geq \frac{2}{3l} a_n > 0. \\ \text{for all } n \geq N.$$

Comparison test now implies the desired.

(2) Similarly,

$$0 < \frac{a_n}{b_n} \leq 1 \quad \text{for } n \gg 1. \quad \underline{\text{Q.E.D.}}$$

The following two tests are based on the comparison with a geometrical series.

Theorem (Root test) Let  $a_n > 0$  ( $n=1, 2, \dots$ ). If there exists  $\alpha$ :  $0 < \alpha < 1$  such that  $\sqrt[n]{a_n} \leq \alpha$  for all  $n \geq n_0$  for some  $n_0$ . Then  $\sum_{n=1}^{\infty} a_n$  converges.

Limit version:  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \alpha < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$  converges.  
(all  $a_n > 0$ )

Proof Let's say  $n_0 = 1$ . So,  $a_n \leq \alpha^n$  ( $n=1, 2, \dots$ )  
~~Since,  $a_n \leq \alpha^n$~~  But  $\sum_{n=1}^{\infty} \alpha^n$  converges. Q.E.D.

Theorem (Ratio Test) Let  $a_n > 0$  ( $n=1, 2, \dots$ ). If there exist  $\alpha \in (-1, 1)$  and  $n_0 \geq 1$  such that

$$\frac{a_{n+1}}{a_n} \leq \alpha \text{ for all } n \geq n_0$$

then  $\sum_{n=1}^{\infty} a_n$  converges.

Limit version:  $a_n > 0$  ( $n=1, 2, \dots$ )  $\Rightarrow \sum_{n=1}^{\infty} a_n$  converges.  
 $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \alpha < 1$ .

Proof. Let's say  $n_0 = 1$ .

$$\frac{a_2}{a_1} \leq \alpha \Rightarrow a_2 \leq \underbrace{a_1 \alpha}_\downarrow$$

$$\frac{a_3}{a_2} \leq \alpha \Rightarrow a_3 \leq a_2 \alpha \leq a_1 \alpha^2$$

$$\frac{a_4}{a_3} \leq \alpha \Rightarrow a_4 \leq a_3 \alpha \leq a_1 \alpha^3$$

$$\dots a_n \leq a_1 \alpha^n. \quad \sum a_1 \alpha^n \text{ converges. } \underline{\text{Q.E.D.}}$$

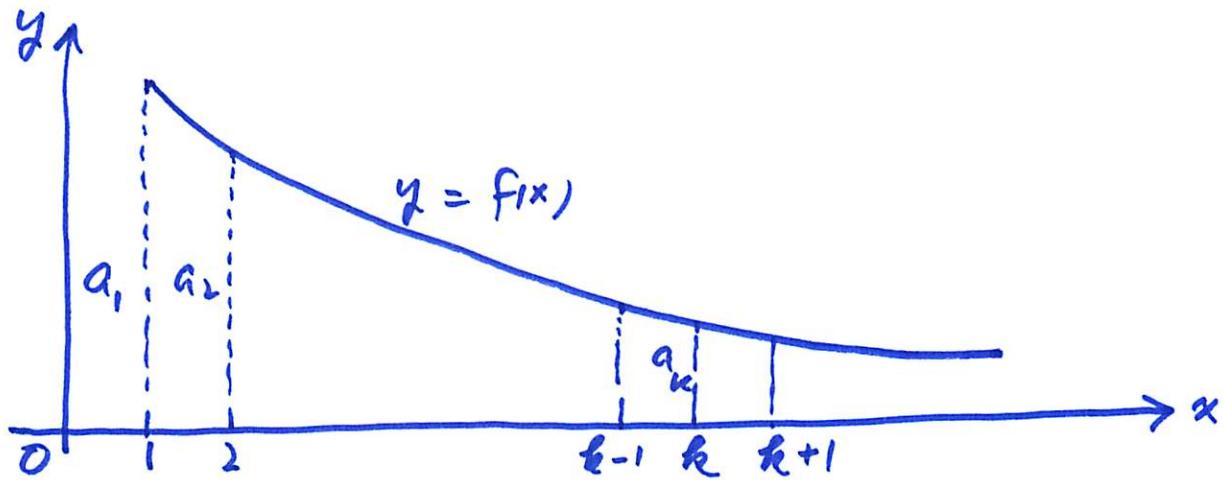
We still have not shown the convergence of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  and divergence of  $\sum_{n=1}^{\infty} \frac{1}{n}$ ! But we are getting there.

Theorem (Integral test) Let  $a_n = f(n)$  ( $n=1, 2, \dots$ ) with  $f: [1, \infty) \rightarrow \mathbb{R}$  a positive, continuous and decreasing function. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x) dx < \infty$$

(i.e.  $\int_1^{\infty} f(x) dx$  converges, i.e.  $\lim_{A \rightarrow +\infty} \int_1^A f(x) dx$  exists.)

Proof



$$\int_k^{k+1} f(t) dt \leq a_k \leq \int_{k-1}^k f(t) dt$$

Sum:  $\sum_{n=1}^{\infty} a_n \leq \int_1^{k+1} f(t) dt \leq \sum_{n=1}^k a_n \leq a_1 + \int_1^k f(t) dt$ .

Q.E.D.

Example  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges  $\iff p > 1$ .

Use the integral test with  $f(x) = (\frac{1}{x})^p$ .

Examples (1)  $\sum_{n=3}^{\infty} \frac{1}{n^p (\ln n)^q}$

$p > 1$ : convergence. compare to  $\frac{1}{n^p}$   
 $p < 1$ : divergence  
 $p = 1$ ,  $q > 1$  convergence  
 $q \leq 1$  divergence

$\frac{1}{n^{p+\varepsilon}} (p+\varepsilon < 1)$

Try  $f(x) = \frac{1}{x(\ln x)^q}$ .

(2)  $\sum_{n=1}^{\infty} \underbrace{\frac{n^2+1}{4n^3+20n-1}}_{= O\left(\frac{1}{n}\right)}$ . divergence

$$\lim_{n \rightarrow \infty} \frac{n^2+1}{4n^3+20n-1} / \frac{1}{4n} = 1.$$

(3)  $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$  diverges!

$$\begin{aligned} S_n &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \dots + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - \sqrt{1} = \sqrt{n+1} - 1 \rightarrow \infty. \end{aligned}$$

(4)  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$  converges.

(5)  $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ .  
Converges!

$$\begin{aligned} 1 - \cos \frac{1}{n} &= 1 - \left(1 - \frac{1}{2n^2}\right) + O\left(\frac{1}{n^4}\right) \\ &= \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right) \end{aligned}$$