

Math 210. Lecture Notes. Bo Li, Fall 2013

## Section 1.2 Infinite Series of Numbers

- ① Alternating Series
- ② Rearrangements
- ③ Product of Two Infinite Series
- ④ Infinite Products

Theorem Let  $a_n \downarrow 0$  (i.e.  $a_0 \geq a_1 \geq a_2 \geq \dots$ ,  $\lim_{n \rightarrow \infty} a_n = 0$ ). Then the alternating series  $\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \dots$  converges. Moreover, for each  $n \geq 0$ .

$$\sum_{n=0}^{2n+1} (-1)^n a_n < a_0 - a_1 + a_2 - a_3 + \dots = \sum_{n=0}^{\infty} (-1)^n a_n < \sum_{n=0}^{2n} (-1)^n a_n.$$

Proof The partial sum is  $S_n = \sum_{k=0}^n (-1)^k a_k$ . ( $n = 0, 1, 2, \dots$ ) Use Cauchy's criterion.

$$|S_{n+p} - S_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \dots + a_{n+p-1} - a_{n+p}$$

if  $p-1$  is odd

$$|S_{n+p} - S_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \dots + a_{n+p-2} - a_{n+p-1} + a_{n+p}$$

if  $p-1$  is even.

First case:  $p-1$  is odd.

$$\begin{aligned} |S_{n+p} - S_n| &= a_{n+1} + (-a_{n+2} + a_{n+3}) + (-a_{n+4} + a_{n+5}) \\ &\quad + \dots + (-a_{n+p-2} + a_{n+p-1}) - a_{n+p} \\ &\leq a_{n+1} - a_{n+p} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Second case:  $p-1$  is even.

$$|S_{n+p} - S_n| \leq a_{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence  $\sum_{n=0}^{\infty} (-1)^n a_n$  converges.

Note that the sign of an alternating series (as a real number) is determined by the very first term of the series. For instance,

$$a_0 - a_1 + a_2 - a_3 + \dots$$

$$= \lim_{n \rightarrow \infty} [(a_0 - a_1) + (a_2 - a_3) + \dots + (a_{2n} - a_{2n+1})] \geq 0.$$

So,

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{2n+1} (-1)^n a_n$$

$$= a_{2n+2} - a_{2n+3} + a_{2n+4} - \dots \geq 0$$

$$\text{Similarly, } \sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{2n} (-1)^n a_n$$

$$= -a_{2n+1} + a_{2n+2} - a_{2n+3} - a_{2n+4} + \dots < 0.$$

Q.E.D.

Examples (1)  $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$  converges. with  $a_n = \frac{1}{2n+1}$ .

But, it is not absolutely convergent.

$$\text{Let } S = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$S - (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}) = \frac{1}{9} - \frac{1}{11} + \dots > 0.$$

$$\text{So, } S > 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \dots$$

(2) Does  $\sum_{n=0}^{\infty} (-1)^n \frac{n-2}{n^2+3n+7}$  converge?

Converge absolutely?

$$a_n = \frac{n-2}{n^2+3n+7} \rightarrow 0. \quad a_n = O\left(\frac{1}{n}\right). \quad \text{So, } \sum (-1)^n a_n$$

does not converge absolutely.

For a finitely many  $a_n$ , they may not decrease.  
Claim that  $a_n \downarrow 0$  for large  $n$ .

Pf Let  $f(n) = a_n = \frac{n-2}{n^2+3n+7}$

$$f'(n) = \frac{n^2+3n+7 - (2n+3)(n-2)}{(n^2+3n+7)^2}$$

$$= \frac{-n^2 + ( )n + ( )}{( )^2} < 0$$

if  $n \geq n_0$  for some  $n_0$ .

Then  $a_n \downarrow (n \geq n_0)$  i.e.,  $a_{n_0} \geq a_{n_0+1} \geq a_{n_0+2} \geq \dots$   
Hence,  $\sum (-1)^n a_n$  converges.

Rearrangements  $\sum_{n=1}^{\infty} a_{\kappa(n)}$  is a rearrangement of  $\sum_{n=1}^{\infty} a_n$   
if  $\kappa \rightarrow n_{\kappa}$  is a 1-1 and onto  
Theorem map from  $\mathbb{N}$  to itself.  $\mathbb{N} = \{1, 2, 3, \dots\}$ .

①  $\sum_{n=1}^{\infty} a_n$  converges absolutely  $\Rightarrow$  any of  
its rearrangement also converges absolutely.

②  $\sum_{n=1}^{\infty} a_n$  converges conditionally (i.e., converges  
but does not converge absolutely)  $\Rightarrow$  There is  
a rearrangement of  $\sum a_n$  that diverges.

### Ideas of Proof

① Cauchy's criterion + absolute convergence.

② The positive part is  $\sum_{n=1}^{\infty} p_n = \infty$   
negative part is  $\sum_{n=1}^{\infty} q_n = -\infty$ .

But  $p_n \rightarrow 0$  and  $N_n \rightarrow 0$ .

$$p_1 + p_2 + \dots + p_{n_1} (> 1)$$

$$p_1 + p_2 + \dots + p_{n_1} + N_1$$

$$p_1 + p_2 + \dots + p_{n_1} + N_1 + p_{n_1+1} + \dots + p_{n_2} (> 2)$$

$$p_1 + p_2 + \dots + p_{n_1} + N_1 + p_{n_1+1} + \dots + p_{n_2} + N_2$$

...  
The rearranged one in fact "converges" to  $+\infty$ .

Q.E.D.

A stronger result is as follows. Suppose  $\sum_{n=1}^{\infty} a_n$  converges conditionally. Suppose  $-\infty < \alpha \leq \beta < \infty$ . Then there exists a rearrangement  $\sum_{n=1}^{\infty} a_{\sigma(n)}$  of  $\sum_{n=1}^{\infty} a_n$ , with partial sums  $S_m = \sum_{n=1}^m a_{\sigma(n)}$ , such that  $\liminf_{m \rightarrow \infty} S_m = \alpha$ ,  $\limsup_{m \rightarrow \infty} S_m = \beta$ .

See, e.g., Rudin's undergrad. analysis book.  
The proof uses the fact that

$$\sum p_n = \infty, \quad \sum N_n = -\infty, \quad p_n \rightarrow 0, \quad N_n \rightarrow 0.$$

## Product of Two Series

How do we define  $(\sum a_n)(\sum b_n)$ ?

If both converge then this is a product of two numbers. But does this number correspond to some series?

A natural definition:

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right)$$

$$= (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + \dots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2$$

$$+ \dots + \underbrace{(a_0 b_0 + a_1 b_1 + \dots + a_n b_0)}_{c_n} x^n + \dots$$

$$c_n = \sum_{k=0}^n a_{n-k} b_k = \sum_{k=0}^n a_k b_{n-k}.$$

Definition

Given  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$ . Let

$$c_n = \sum_{k=0}^n a_k b_{n-k} \quad (n=0, 1, 2, \dots)$$

Call  $\sum_{n=0}^{\infty} c_n$  the product series of  $\sum_{n=0}^{\infty} a_n$  and

$$\sum_{n=0}^{\infty} b_n.$$

Theorem (1) If  $\sum_{n=0}^{\infty} a_n$  and converges absolutely,  $\sum_{n=0}^{\infty} b_n$  converges. Then  $\sum_{n=0}^{\infty} c_n$  ( $c_n = \sum_{k=0}^n a_k b_{n-k}$ ) converges. Moreover,  $\sum_{n=0}^{\infty} c_n = \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right)$ .

(2) If  $\sum_{n=0}^{\infty} a_n = A$ ,  $\sum_{n=0}^{\infty} b_n = B$ , and  $\sum_{n=0}^{\infty} c_n = C$  ( $c_n = \sum_{k=0}^n a_k b_{n-k}$ ) all converge, then  $\left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) = \left( \sum_{n=0}^{\infty} c_n \right)$ .

Example.  $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$  ( $n=0, 1, 2, \dots$ ).  $\sum_{n=0}^{\infty} a_n$  converges  
 $\Rightarrow c_n = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}}$

$$\text{so } n \leq k \leq n: (n-k+1)(k+1) = nk+n-k^2-k+k+1 = nk+n-k^2+1$$

$$\leq n^2+n+1 \leq (n+1)^2$$

$$\text{So, } |c_n| \geq \sum_{k=0}^n \frac{1}{\sqrt{n+1}} = 1. \quad c_n \not\rightarrow 0. \quad \sum c_n \text{ diverges.}$$

## Infinite Product

Consider  $a_n \in \mathbb{C}$  ( $n=1, 2, \dots$ ). Assume each  $a_n \neq 0$  ( $n=1, 2, \dots$ ). Let  $p_n = \prod_{k=1}^n a_k = a_1 \cdot a_2 \cdot \dots \cdot a_n$ .

Definition If  $\lim_{n \rightarrow \infty} p_n = p$  exists and  $p \neq 0$  then  $\prod_{n=1}^{\infty} a_n$  converges and  $\prod_{n=1}^{\infty} a_n = p$ . Otherwise, we say  $\prod_{n=1}^{\infty} a_n$  diverges. (Even  $\lim_{n \rightarrow \infty} p_n = 0$  exists, we still say  $\prod_{n=1}^{\infty} a_n$  diverges to 0.).

## Discussions

(1) If  $p_n \rightarrow p$  then  $|p_n| \rightarrow |p|$ . Hence, if  $\prod_{n=1}^{\infty} |a_n|$  converges, then  $\prod_{n=1}^{\infty} |a_n|$  converges and  $\prod_{n=1}^{\infty} |a_n| = |\prod_{n=1}^{\infty} a_n|$ .

(2) Assume  $\prod_{n=1}^{\infty} |a_n|$  converges. Then  $\prod_{n=1}^{\infty} |a_n|$  converges. Hence

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n |a_k| = p \quad (\neq 0) \text{ exists.}$$

$$\text{But } \prod_{k=1}^n |a_k| = e^{\sum_{k=1}^n \ln |a_k|} = e^{\sum_{k=1}^n \ln |a_k|}$$

$$p = e^{\sum_{k=1}^n \ln |a_k|}$$

$$\text{Hence, } \sum_{k=1}^n \ln |a_k| \rightarrow \ln p.$$

i.e.,  $\sum_{k=1}^{\infty} \ln |a_k|$  converges.

Hence  $\ln |a_n| \rightarrow 0$ . i.e.,  $|a_n| \rightarrow 1$ .

(3) In fact, if  $\prod_{n=1}^{\infty} a_n$  converges, then we must have  $a_n \rightarrow 1$  as  $n \rightarrow \infty$ .

Let's consider all  $a_n \in \mathbb{R}$  (real numbers). If there exists a subseq of  $a_n$  that ~~converges~~ converges to  $-1$ , then  $\{p_n\}$  has a subsequence that alternates in sign.  $p_{n_1} > 0, p_{n_2} < 0, p_{n_3} > 0, p_{n_4} < 0, \dots$ . But  $|p_{n_k}| \rightarrow |p| \neq 0$ . Thus, we won't have  $p_{n_k} \rightarrow p$  as  $k \rightarrow \infty$ . This means  $\prod_{n=1}^{\infty} a_n$  does not converge,  $\therefore$  contradiction.

We have proved:

or complex

Theorem If all  $a_n$  are real numbers,  $a_n \neq 0$  ( $n = 1, 2, \dots$ ), and  $\prod_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 1$ .

Pf. Let  $p_n = \prod_{k=1}^n a_k \rightarrow p \neq 0$ .  $a_k = r_k e^{i\theta_k}$ ,  $p = |p|e^{i\theta}$ .  $\begin{cases} r_k \rightarrow 1 \\ \frac{1}{r_k} e^{i\theta_k} \rightarrow e^{i\theta} \end{cases} \Rightarrow$   
 $\text{If } a_n \approx 1 \text{ then } a_n = 1 + b_n \text{ with } b_n \approx 0$ .  
 $\text{So, we consider } \prod_{n=1}^{\infty} (1 + b_n) \text{ or } \prod_{n=1}^{\infty} (1 - b_n)$

$\begin{cases} e^{i\theta n+1} \rightarrow 1 \\ a_{n+1} \rightarrow 1 \end{cases} \quad Q.E.D.$

Theorem (1) Assume  $b_n > 0$  ( $n = 1, 2, \dots$ ). Then  $\prod_{n=1}^{\infty} (1 + b_n)$  converges  $\Leftrightarrow \sum_{n=1}^{\infty} b_n$  converges.

(2) Assume  $0 < b_n < 1$  ( $n = 1, 2, \dots$ ). Then  $\prod_{n=1}^{\infty} (1 - b_n)$  converges  $\Leftrightarrow \sum_{n=1}^{\infty} b_n$  converges.

Proof (1) All  $b_n > 0$  ( $n = 1, 2, \dots$ ).  $p_n = \prod_{k=1}^n (1 + b_k)$ .

$$S_n = \sum_{k=1}^n b_k.$$

$$\begin{aligned} 0 < b_1 < \sum_{k=1}^n b_k = S_n &< \prod_{k=1}^n (1 + b_k) = e^{\sum_{k=1}^n \ln(1 + b_k)} \\ &< e^{\sum_{k=1}^n b_k} = e^{S_n} \end{aligned}$$

We used:  $\sum_{k=1}^n b_k < \prod_{k=1}^n (1+b_k)$

$$\text{e.g., } (1+b_1)(1+b_2)(1+b_3)$$

$$= 1 + (b_1 + b_2 + b_3) + (b_1 b_2 + b_1 b_3 + b_2 b_3) + b_1 b_2 b_3$$

$$> b_1 + b_2 + b_3$$

Also:  $\ln(1+x) < x \text{ if } 0 < x < 1. \text{ (or if fact)}$   
if  $x > 0$

$$S_n < p_n < e^{S_n} \quad (n=1, 2, \dots)$$

$\{p_n\}$  converges  $\Rightarrow$  bounded  $\Rightarrow \{S_n\}$  bounded  
 $\Rightarrow \{S_n\}$  converges.

$\{S_n\}$  converges  $\Rightarrow \{p_n\}$  bounded. But  $p_n \uparrow$   
 So,  $\{p_n\}$  converges to some  $p$ . Clearly,  $p \geq 1$ .

(2)  $\prod_{k=1}^{\infty} (1-b_k)$  converges  $\Rightarrow 1-b_k \rightarrow 1 \iff b_k \rightarrow 0$   
~~if~~  $\sum b_k$  converges  $\Rightarrow b_k \rightarrow 0$ .

So, assume  $\lim_{k \rightarrow \infty} b_k = 0$ .

$\prod_{k=1}^n (1-b_k)$  converges  $\Leftrightarrow \sum_{k=1}^n \ln(1-b_k)$  converges.

but  $\frac{\ln(1-b_k)}{-b_k} \rightarrow 1$  as  $k \rightarrow \infty$  (since  $b_k \rightarrow 0^+$ )

Thus,  $\sum_{k=1}^{\infty} \ln(1-b_k)$  converges  $\Leftrightarrow \sum_{k=1}^{\infty} b_k$  converges.

Q.E.D.

Example  $\prod_{n=1}^{\infty} \frac{n^3-1}{n^3+1} = \prod_{n=1}^{\infty} \frac{n^3+1-2}{n^3+1} = \prod_{n=1}^{\infty} \left(1 - \frac{2}{n^3+1}\right)$   
 converges, since  $\sum \frac{2}{n^3+1}$  converges.