

# Math 210. Lecture Notes. Bo Li, Fall 2013

- sequences of numbers
- infinite series of numbers
- sequences of functions
- infinite series of functions
- power series

## Sequences of Functions

The set of all real numbers.

Let  $I$  be an interval of  $\mathbb{R}$  (open, closed, half open, bounded, unbounded, etc.).

Consider functions  $f, f_n : I \rightarrow \mathbb{R}$  ( $n=1, 2, \dots$ )

Definition  $\{f_n\}$  converges to  $f$  on  $I$  pointwise, denoted  $f_n \rightarrow f$  on  $I$  pointwise, if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in I$$

( $\forall$  means for all, for any, etc.)

Examples (1)  $I = [0, 2\pi]$ .  $f_n(x) = \frac{\sin nx}{\sqrt{n}}$   
 $f(x) = 0$ .  $\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0$  for any  $x$ .

Is it true that  $\lim_{n \rightarrow \infty} f_n'(x) = f'(x)$ ?

No!  $f_n'(x) = \sqrt{n} \cos nx$ .  $f'(x) = 0$ .

For  $x=0$ ,  $f_n'(0) = \sqrt{n} \rightarrow +\infty$ .

In fact,  $\{f_n'(x)\} = \{\sqrt{n} \cos nx\}$  does not converge for any  $x$ . Otherwise, fix  $x$ ,  $\cos nx \rightarrow 0$ . Then  $\cos(2nx) \rightarrow 0$ . But  $\cos(2nx) = 2\cos^2(nx) - 1 \rightarrow -1$ . This is a contradiction!

$$(2) \quad g_n(x) = n^2 x (1-x^2)^n \quad (n=1, 2, \dots), \quad x \in [0, 1]$$

(Notation:  $\in$  means "belongs to")

$$g_n(0) = 0 \quad g_n(1) = 0 \quad \text{for all } n \geq 1.$$

If  $0 < x < 1$ , then  $g_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  (why?)

Is it true that

$$\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx = \int_0^1 \lim_{n \rightarrow \infty} g_n(x) dx ?$$

No! The right-hand side is 0. But the left-hand side is not:

$$\begin{aligned} \int_0^1 g_n(x) dx &= n^2 \int_0^1 x (1-x^n)^n dx \\ &\stackrel{x^n=y}{=} n^2 \int_0^1 (1-y)^n \frac{1}{n} dy = \frac{n^2}{n(n+1)} \rightarrow +\infty ! \end{aligned}$$

$$(3) \quad f_0(x) = 1, \quad f_1(x) = 1+x, \quad f_2(x) = 1+x+\frac{x^2}{2!}, \dots$$

$$f_n(x) = 1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}$$

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) = e^x \quad \text{for all } x \in \mathbb{R}.$$

Definition  $\{f_n\}$  converges to  $f$  uniformly on  $I$ , denoted  $f_n \rightarrow f$  uniformly on  $I$ , if  $\forall \varepsilon > 0, \exists N = N(\varepsilon)$  such that  $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$  for all  $x \in I$ .

Remark. For a fixed  $x$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  is defined as follows:  $\forall \varepsilon > 0, \exists N = N(\varepsilon, x)$  s.t.  $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$ .

So,  $N$  depends on  $\varepsilon$  and  $x$ .

Uniform convergence:  $N$  does not dep. on  $x$ . i.e., there is an  $N$  that works for all  $x$ .

Remark unif. convergence  $\implies$  pointwise convergence.

Theorem (1)  $f_n \rightarrow f$  uniformly on  $I$   
 $\iff \sup_{x \in I} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$

[For each  $n$ ,  $\sup_{x \in I} |f_n(x) - f(x)|$  is a number.]

(2)  $\{f_n\}$  converges on  $I$  uniformly

$\iff \forall \varepsilon > 0, \exists N = N(\varepsilon)$ . s.t.

$n, m \geq N \implies |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in I$ .

[Cauchy criterion]

Example  $f_n(x) = x^n$  ( $n=1, 2, \dots$ ) .

Case 1.  $I = [0, \alpha]$ . ( $0 < \alpha < 1$ , fixed)

If  $x \in I$ , i.e.,  $0 \leq x \leq \alpha$ , then  $f_n(x) = x^n \rightarrow 0$  (since  $0 < \alpha < 1$ ). So,  $f(x) = 0$ .

$f_n \rightarrow f$  on  $[0, \alpha]$  pointwise.

Claim  $f_n \rightarrow f$  on  $[0, \alpha]$  uniformly.

Pf  $\sup_{x \in I} |f_n(x) - f(x)| = \alpha^n \rightarrow 0$  as  $n \rightarrow \infty$ .

Case 2  $I = [0, 1]$ .  $f_n(x) \rightarrow f(x) = 0$  as  $n \rightarrow \infty$  for any  $x \in I$ , i.e.,  $0 \leq x \leq 1$ .

Claim  $f_n \not\rightarrow f$  uniformly on  $I = [0, 1]$

Pf For each  $n \geq 1$ ,

$$\sup_{x \in I} |f_n(x) - f(x)| = \sup_{0 \leq x \leq 1} |f_n(x) - f(x)|$$

$$= \sup_{0 \leq x \leq 1} x^n \geq \left(\frac{1}{\sqrt[2]{2}}\right)^n = \frac{1}{2}$$

choose  $x_n = \frac{1}{\sqrt[2]{2}} \in [0, 1]$

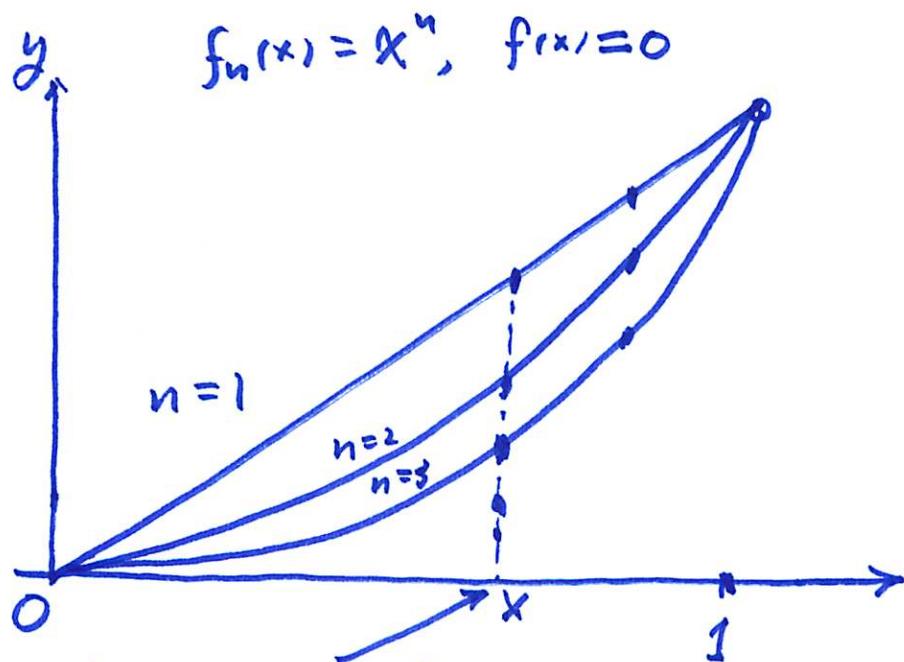
$\sup$  over all  $x \in [0, 1]$  is

bigger than the value at  $x_n$ .

Hence.  $\sup_{x \in I} |f_n(x) - f(x)| \not\rightarrow 0$ . Q.E.D.

end of proof

Note: In fact,  $\sup_{x \in I} |f_n(x) - f(x)| = 1$ .



at each  $x$ ,  $f_n(x) \rightarrow 0$  as  $n \rightarrow \infty$

but the convergence is not uniform  
for all  $x$  in  $[0, 1]$ .  $f_n(x) \rightarrow 0$   
is slower for  $x$  closer to 1.

Theorem (1)  $f_n \rightarrow f$  unif. on  $I$ .

$$\lim_{x \rightarrow x_0} f_n(x) = A_n \quad (n=1, 2, \dots)$$

$\Rightarrow \{A_n\}$  converges, and

$$\lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n(x).$$

(2)  $f_n \rightarrow f$  on  $I = [a, b]$  unif.

All  $f_n$  are integrable on  $I$ .

$\Rightarrow f$  is also integrable on  $I$ , and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b [\lim_{n \rightarrow \infty} f_n(x)] dx.$$

(3)  $f_n' \rightarrow f'$  unif. on  $I = [a, b]$   
 $f_n(x_0) \rightarrow f(x_0)$  for some  $x_0 \in [a, b]$ .

$\Rightarrow f_n \rightarrow f$  on  $I$  unif. and

$$\frac{d}{dx} \left( \lim_{n \rightarrow \infty} f_n(x) \right) = \lim_{n \rightarrow \infty} \frac{df_n(x)}{dx} \quad \forall x \in [a, b].$$

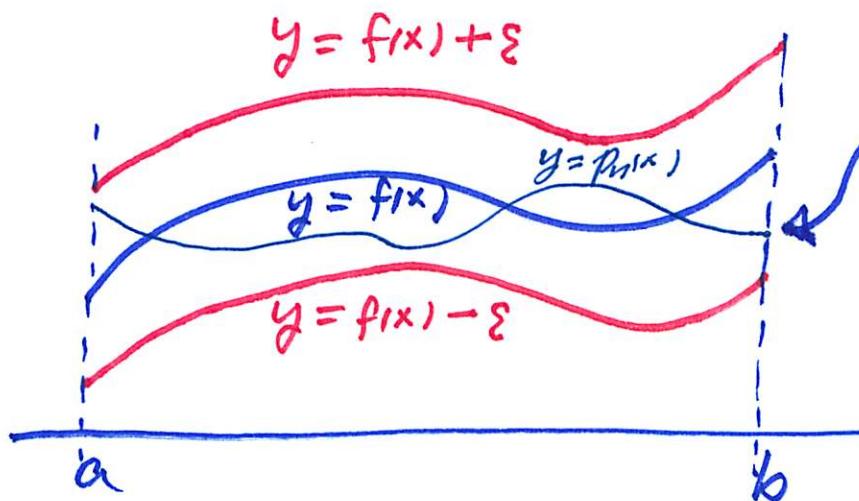
The Weierstrass Theorem Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function. There exist polynomials  $p_1(x), p_2(x), \dots, p_n(x), \dots$  such that  $p_n \rightarrow f$  on  $[a, b]$  uniformly.

Remark  $p_n \rightarrow f$  on  $[a, b]$  uniformly :  $\forall \varepsilon > 0$  there exists  $N = N(\varepsilon) > 0$ , such that

$$n \geq N \implies |p_n(x) - f(x)| < \varepsilon \text{ for all } x \in [a, b]$$

i.e.

$$f(x) - \varepsilon < p_n(x) < f(x) + \varepsilon \text{ for all } x.$$



If  $n \geq N$ , then  
the graph of  
 $p_n(x)$  lies in  
between the  
two curves  
 $\rightarrow$   
of  $y = f(x) + \varepsilon$   
and  $y = f(x) - \varepsilon$ .

Bernstein polynomials.  $I = [0, 1]$ ,  $f: [0, 1] \rightarrow \mathbb{R}$  continuous.

$$P_n(x) = B_n(f; x) = \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Theorem  $B_n(f; x) \rightarrow f(x)$  unif. on  $I = [0, 1]$ .

Example Let  $u \in C([0, 1])$ . This means  $u = u(x)$  is a continuous function on  $[0, 1]$ . Suppose

$$\int_0^1 x^n u(x) dx = 0 \quad (n = 0, 1, 2, \dots).$$

Prove that  $u(x) = 0$  on  $[0, 1]$ .

Proof. Let  $\{P_k\}$  be a sequence of polynomials that converge to  $f$  unif. on  $[0, 1]$ . (By the Weierstrass Theorem, such  $P_k$ 's exist.)

The assumption implies that

$$\int_0^1 P_k(x) u(x) dx = 0 \quad (k = 1, 2, \dots)$$

(why?).

$$\begin{aligned} \text{Hence, } \int_0^1 u(x)^2 dx &= \int_0^1 [u(x)^2 - P_k(x) u(x)] dx \\ &= \int_0^1 [u(x) - P_k(x)] u(x) dx \leq \int_0^1 \left[ \sup_{x \in [0, 1]} |u(x) - P_k(x)| \right] \\ &\cdot |u(x)| dx = \sup_{x \in [0, 1]} |u(x) - P_k(x)| \int_0^1 |u(x)| dx \\ &\longrightarrow 0 \text{ as } k \rightarrow \infty. \text{ Hence } u \equiv 0. \quad \underline{\text{Q.E.D.}} \end{aligned}$$