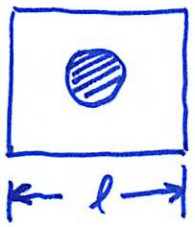
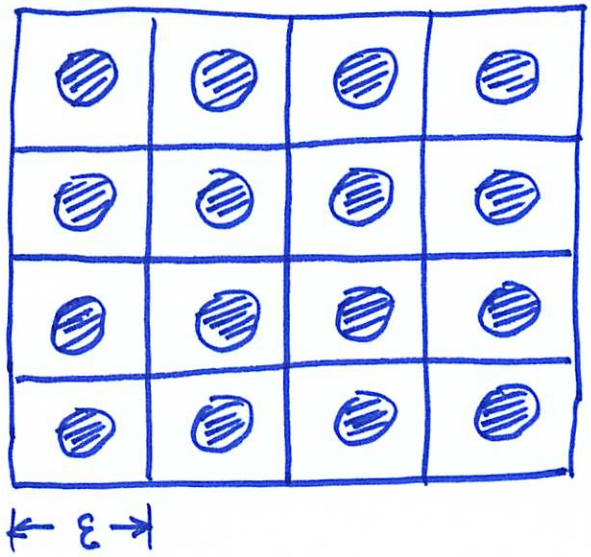
- Some more results related to weak/strong convergence with application
- Dirac δ-functions

<u>Example</u> Composite material.





mixture of two different material. (or two different phases of a material)

One-dimensional: Let $a = a(x)$ be an ℓ-periodic function. Let $a_k(x) = a(kx)$, $(k = 1, 2, \cdots)$. [For $k \gg 1$, a_k represents a composite material in terms of material constants, e.g., elastic moduli, heat conductivity, dielectrics, etc.] Define

$$\bar{a} = \frac{1}{\ell}\int_0^\ell a(x)\,dx.$$

This is the average of a on a unit cell $[0, \ell]$.

<u>Theorem</u> $a_k \longrightarrow \bar{a}$ weakly in $L^2(0, L)$ for $L > 0$.

[Here, L is material size.]

Note. $\bar{a}$ is Not the effective material constant.

<u>Example</u> Consider elastic energy functional

$$I[u] = \int_0^1 \left[(u'(x)^2 - 1)^2 + u(x)^2 \right] dx.$$

Clearly, $I[u] \geq 0$ for all functions (displacements) $u : [0,1] \to \mathbb{R}$. If $I[u]$ is small, then $u'(x) \approx \pm 1$ and $u(x) \approx 0$.

Definition of $u_k : [0,1] \to 1$

The energy of u_k is

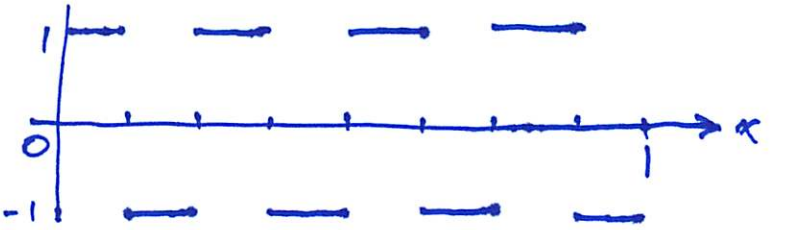
$$I[u_k] = \int_0^1 u_k(x)^2 dx \leq \left(\frac{1}{2k}\right)^2 \to 0 \quad \text{as } k \to \infty.$$

So, $\inf_u I[u] = 0$.

But, there exists no u such that $I[u] = 0$.

<u>Proof</u> If $I[u] = 0$. Then $\int_0^1 u(x)^2 dx = 0$. Hence $u(x) \equiv 0$. Thus $u'(x) \equiv 0$. and $I[u] = 1$. Contradiction!

Now, let $f_k(x) = u_k'(x)$



Then, $f_k \to 0$ weakly in $L^2(0,1)$.

$L^2(a,b)$ is an infinitely dimensional space. (We will learn this later.) Such a space is very different from a finitely dimensional one.

<u>Theorem</u> Every bounded sequence of real numbers has a subsequence that converges.

<u>Theorem</u> Every bounded sequence in $L^2(a,b)$ has a subsequence that converges weakly in $L^2(a,b)$.

<u>Remark</u> Here, a sequence $\{u_k\}$ that is bounded in $L^2(a,b)$ means the numeric sequence

$$\|u_k\|_{L^2(a,b)} = \sqrt{\int_a^b |u_k(x)|^2 dx} \quad (k=1,2,\cdots)$$

is bounded.

Dirac δ-function

A very useful concept for modeling point forces, point charges, particle densities, etc.

Sometimes the δ-function $\delta = \delta(x)$ is defined as:

(1) $\delta(x) = 0$ if $x \neq 0$

(2) $\delta(0) = \infty$

(3) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$

But there is no such function — these defining conditions contradict:

$$1 \overset{(3)}{=} \int_{-\infty}^{\infty} \delta(x)\,dx = \lim_{\substack{\varepsilon \to 0^+ \\ A \to \infty}} \left(\int_{-A}^{-\varepsilon} \delta(x)\,dx + \int_{\varepsilon}^{A} \delta(x)\,dx \right)$$

def. of improper integral.

$$\overset{(1)}{=} \lim_{\substack{\varepsilon \to 0^+ \\ A \to \infty}} \left(\int_{-A}^{-\varepsilon} 0\,dx + \int_{\varepsilon}^{A} 0\,dx \right)$$

$$= \lim_{\substack{\varepsilon \to 0^+ \\ A \to \infty}} (0 + 0)$$

$$= 0 \qquad 1 = 0 \text{ ! A contradiction!}$$

<u>Definition</u> The Dirac δ-function (at 0) is a <u>functional</u> on the class of all continuous functions:

$$\delta(\phi) = \phi(0) \qquad \phi \text{ is any cont. function.}$$

Sometimes it is written as

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

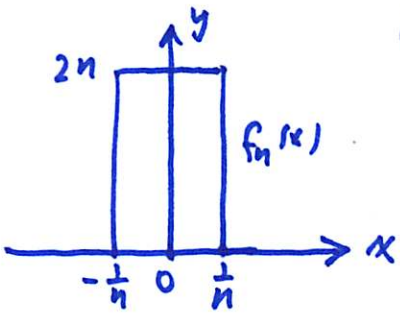
<u>Example</u> $\phi(x) = e^x$ $\quad \delta(\phi) = \int_{-\infty}^{\infty} \delta(x)\, e^x\, dx = e^0 = 1$

<u>Property</u> (1) δ is linear:

$$\delta(\alpha\phi + \beta\psi) = \alpha\,\delta(\phi) + \beta\,\delta(\psi) = \alpha\,\phi(0) + \beta\,\psi(0)$$

α, β: numbers. ϕ, ψ: cont. functions.

(2) δ is continuous: if $\phi_n \to \phi$ pointwise (in a neighborhood of 0, or just $\phi_n(0) \to \phi(0)$), then $\delta(\phi_n) \to \delta(\phi)$. i.e., $\phi_n(0) \to \phi(0)$.



(3) Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2n & \text{if } -\frac{1}{n} \le x \le \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \text{ or } x < -\frac{1}{n} \end{cases}$$

Then $f_n \to \delta$ weakly in the sense that $\int_{-\infty}^{\infty} f_n(x)\,\phi(x)\,dx \longrightarrow \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx$

i.e., $\int_{-\infty}^{\infty} f_n(x)\,\phi(x)\,dx \longrightarrow \phi(0)$

for any continuous function ϕ.

<u>Proof of (3)</u> $\left| \int_{-\infty}^{\infty} f_n(x)\,\phi(x)\,dx - \phi(0) \right|$

$$= \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n\,\phi(x)\,dx - \int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(0)\,2n\,dx \right|$$

$$= \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n\,[\phi(x) - \phi(0)]\,dx \right|$$

$$\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \left| \phi(x) - \phi(0) \right| dx$$

$$\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \left(\max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} \left| \phi(x) - \phi(0) \right| \right) dx$$

$$= \left(\max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} \left| \phi(x) - \phi(0) \right| \right) \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n\, dx$$

$$= \max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} \left| \phi(x) - \phi(0) \right|$$

$\longrightarrow 0$ as $n \to \infty$ (since ϕ is cont. at 0).

<u>Q.E.D.</u>

In fact, (3) can generalized to more general f_n.

Suppose $f_n : \mathbb{R} \to \mathbb{R}$ satisfy:

(1) $f_n \geq 0$ in $\mathbb{R}$.

$f_n = 0$ outside $[-\varepsilon_n, \varepsilon_n]$ with $\varepsilon_n \downarrow 0$.

(2) $\int_{-\infty}^{\infty} f_n\, dx = \int_{-\varepsilon_n}^{\varepsilon_n} f_n\, dx = 1$ exists for each $n \geq 1$.

Then $f_n \to \delta$ weakly.

<u>Pf</u> $\left| \int_{-\infty}^{\infty} f_n(x)\, \phi(x)\, dx - \phi(0) \right|$

$$= \left| \int_{-\varepsilon_n}^{\varepsilon_n} f_n(x) \left[\phi(x) - \phi(0) \right] dx \right|$$

$$\leq \int_{-\varepsilon_n}^{\varepsilon_n} f_n(x) \left| \phi(x) - \phi(0) \right| dx$$

$$\leq \sup_{-\varepsilon_n \leq x \leq \varepsilon} \left| \phi(x) - \phi(0) \right| \underbrace{\int_{-\varepsilon_n}^{\varepsilon_n} f_n(x)\, dx}_{=1 \text{ by assumption (2)}}$$

$\longrightarrow 0$ as $n \to \infty$

if ϕ is cont. at 0. <u>Q.E.D.</u>

Derivatives of δ- function

$$\int_{-\infty}^{\infty} \delta'(x)\, \phi(x)\, dx = -\int_{-\infty}^{\infty} \delta(x)\, \phi'(x)\, dx = -\phi'(0)$$

Define δ' to be the functional on continuously differentiable functions ϕ:

$$\delta'(\phi) = -\delta(\phi') = -\phi'(0)$$

Similarly

$$\delta''(\phi) = -\delta'(\phi') = \delta(\phi'') = \phi''(0)$$

$$\delta^{(k)}(\phi) = (-1)^k \phi^{(k)}(0).$$

$\delta^{(k)}$ is linear and continuous: $\phi_n^{(k)} \to \phi^{(k)}$

$\Rightarrow \delta^{(k)}(\phi_n) \to \delta^{(k)}(\phi)$. as $n \to \infty$.

<u>δ-function at x_0</u> δ_{x_0} or $\delta(x - x_0)$

<u>Definition</u> $\delta_{x_0}(\phi) = \phi(x_0)$ for any cont. ϕ.

<u>Example</u> Point charges in a biomolecule (e.g., a DNA, a protein, etc.)

$$\rho(x) = \sum_{i=1}^{N} Q_i\, \delta_{x_i}(x)$$

charge density

partial charge carried by the ith particle

location of the ith particle