Some more results related to weak/strong convergence with application

Example: Composite material.

One-dimensional: Let \( a = a(x) \) be an \( l \)-periodic function. Let \( a_k(x) = a(kx) \), \( (k = 1, 2, \ldots) \). [For \( k \gg 1 \), \( a_k \) represents a composite material in terms of material constants, e.g., elastic moduli, heat conductivity, dielectrics, etc.] Define

\[
\overline{a} = \frac{1}{l} \int_0^l a(x) \, dx.
\]

This is the average of \( a \) on a unit cell \([0, l]\).

Theorem: \( a_k \rightarrow \overline{a} \) weakly in \( L^2(0,L) \) for \( L > 0 \). [Here, \( L \) is material size.]

Note: \( \overline{a} \) is not the effective material constant.
Example: Consider the elastic energy functional
\[ I[u] = \int_0^1 \left[ (u''(x))^2 + 1 \right] \, dx. \]

Clearly, \( I[u] \geq 0 \) for all functions (displacement) \( u : [0, 1] \rightarrow \mathbb{R} \). If \( I[u] \) is small, then \( u'(x) \equiv \pm 1 \) and \( u(x) \equiv 0 \).

The energy of \( u_k \) is
\[ I[u_k] = \int_0^1 u_k(x)^2 \, dx \leq \left( \frac{1}{2k} \right)^2 \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

So, \( \inf_u I[u] = 0 \).

But, there exists no \( u \) such that \( I[u] = 0 \).

Proof: If \( I[u] = 0 \), then \( \int_0^1 u(x)^2 \, dx = 0 \). Hence \( u(x) \equiv 0 \). Thus \( u'(x) \equiv 0 \) and \( I[u] = 1 \). Contradiction!

Now, let \( f_k(x) = u_k'(x) \)

Then, \( f_k \rightarrow 0 \) weakly in \( L^2(0, 1) \).
$L^2(a,b)$ is an infinitely dimensional space. (We will learn this later.) Such a space is very different from a finitely dimensional one.

**Theorem** Every bounded sequence of real numbers has a subsequence that converges.

**Theorem** Every bounded sequence in $L^2(a,b)$ has a subsequence that converges weakly in $L^2(a,b)$.

**Remark** Here, a sequence $\{u_n\}$ that is bounded in $L^2(a,b)$ means the numeric sequence $\|u_n\|_{L^2(a,b)} = \sqrt{\int_a^b |u_n(x)|^2 \, dx}$ (\(k=1, 2, \ldots\)) is bounded.
Dirac $\delta$-function

A very useful concept for modeling point forces, point charges, particle densities, etc.

Sometimes the $\delta$-function $\delta = \delta(x)$ is defined as:
1. $\delta(x) = 0$ if $x \neq 0$
2. $\delta(0) = \infty$
3. $\int_{-\infty}^{\infty} \delta(x) dx = 1.$

But there is no such function — these defining conditions contradict:

$$\int_{-\infty}^{\infty} \delta(x) dx = \lim_{\varepsilon \to 0+} \lim_{A \to \infty} \left( \int_{\varepsilon}^{A} \delta(x) dx + \int_{-\infty}^{\varepsilon} \delta(x) dx \right)$$

def. of improper integral.

$$\lim_{\varepsilon \to 0+} \lim_{A \to \infty} \left( \int_{\varepsilon}^{A} 0 dx + \int_{-\infty}^{\varepsilon} 0 dx \right) = 0 \quad \vdots \quad 1 = 0 ! A \text{ contradiction!}$$

**Definition.** The Dirac $\delta$-function (at $0$) is a functional on the class of all continuous functions:

$$\delta(\phi) = \phi(0) \quad \phi \text{ is a cont. function.}$$
Sometimes it is written as
\[ \sum_{n=1}^{\infty} d_n(x) \phi(x) \, dx = \phi(0). \]

**Example**
\[ \phi(x) = e^x \quad d_n(x) = \int_{-\infty}^{\infty} e^n x \, dx = e^n = 1 \]

**Property**
1. \( d \) is linear:
\[ d(\alpha \phi + \beta \psi) = \alpha d(\phi) + \beta d(\psi) = \alpha \phi(0) + \beta \psi(0) \]
\( \alpha, \beta \) : numbers, \( \phi, \psi \) : cont. functions.

2. \( d \) is continuous: if \( \phi_n \to \phi \) pointwise
   (in a neighborhood of 0, or just \( \phi_n(0) \to \phi(0) \)),
   then \( d(\phi_n) \to d(\phi) \), i.e., \( \phi_n(0) \to \phi(0) \).

3. Define \( f_n : \mathbb{R} \to \mathbb{R} \) by
\[ f_n(x) = \begin{cases} 2n & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \text{ or } x < -\frac{1}{n} \end{cases} \]

Then \( f_n \to d \) weakly in the sense that
\[ \sum_{n=1}^{\infty} f_n(x) \phi(x) \, dx \to \sum_{n=1}^{\infty} d(x) \phi(x) \, dx \]
\( i.e., \sum_{n=1}^{\infty} f_n(x) \phi(x) \, dx \to \phi(0) \)
for any continuous function \( \phi \).

**Proof of (3)**
\[ \left| \sum_{n=1}^{\infty} f_n(x) \phi(x) \, dx - \phi(0) \right| = \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \phi(x) \, dx - \int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(0) \, dx \right| 
= \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \left[ \phi(x) - \phi(0) \right] \, dx \right| \]
\[\begin{align*}
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} |\phi(x) - \phi(0)| \, dx \\
&= \left( \max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} |\phi(x) - \phi(0)| \right) \int_{-\frac{1}{n}}^{\frac{1}{n}} 2n \, dx \\
&= \max_{-\frac{1}{n} \leq x \leq \frac{1}{n}} |\phi(x) - \phi(0)| \\
&\rightarrow 0 \text{ as } n \rightarrow \infty \quad \text{(since } \phi \text{ is cont. at } 0). \quad \text{Q.E.D.}
\end{align*}\]

In fact, (3) can be generalized to more general \( f_n \).

Suppose \( f_n : \mathbb{R} \rightarrow \mathbb{R} \) satisfy:

1. \( f_n \geq 0 \) in \( \mathbb{R} \).
   \( f_n = 0 \) outside \([-\xi_n, \xi_n]\) with \( \xi_n \downarrow 0 \).
2. \( \int_{-\xi_n}^{\xi_n} f_n \, dx = \sum_{n} f_n \, dx = 1 \) exists for each \( n \geq 1 \).

Then \( f_n \rightarrow f \) weakly.

\[\begin{align*}
\Phi_n &= \int_{-\xi_n}^{\xi_n} \Phi(x) - \phi(0) \, dx \\
&= \left[ \sum_{n} \int_{-\xi_n}^{\xi_n} f_n(x) \Phi(x) \, dx - \phi(0) \right] \\
&\leq \sum_{n} \int_{-\xi_n}^{\xi_n} f_n(x) \Phi(x) \, dx - \phi(0) \\
&\leq \sup_{-\xi_n \leq x \leq \xi_n} |\phi(x) - \phi(0)| \sum_{n} \int_{-\xi_n}^{\xi_n} f_n(x) \, dx \\
&\leq \sum_{n} \int_{-\xi_n}^{\xi_n} f_n(x) \, dx \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{if } \phi \text{ is cont. at } 0. \quad \text{Q.E.D.}
\end{align*}\]
Derivatives of \( \delta \) function

\[ \int_{-\infty}^{\infty} \delta'(x) \phi(x) \, dx = -\int_{-\infty}^{\infty} \delta''(x) \phi(x) \, dx = -\phi'(0) \]

Define \( \delta' \) to be the functional on continuously differentiable functions \( \phi \):

\[ \delta'(\phi) = -\delta(\phi') = -\phi'(0) \]

Similarly

\[ \delta''(\phi) = -\delta'(\phi') = \delta(\phi'') = \phi''(0) \]

\[ \delta''(\phi) = (-1)^{k-1} \phi^{(k)}(0). \]

\( \delta^{(k)} \) is linear and continuous:

\[ \phi_n \to \phi \]

\[ \delta^{(k)}(\phi_n) \to \delta^{(k)}(\phi) \text{ as } n \to \infty. \]

\( \delta \)-function at \( x_0 \): \( \delta_{x_0} \) or \( \delta(x-x_0) \)

Definition \( \delta_{x_0}(\phi) = \phi(x_0) \) for any cont. \( \phi \).

Example: Point charges in a biomolecule (e.g., a DNA, a protein, etc.)

\[ p(x) = \sum_{i=1}^{N} Q_i \delta_{x_i}(x) \]

\( p(x) \) = charge density

location of the \( i \)th particle

partial charge carried by the \( i \)th particle