

We now introduce two more types of convergence of sequences of functions.

Notation

$$L^1(a, b) = \left\{ f: [a, b] \rightarrow \mathbb{R} : \int_a^b |f| dx < \infty \right\}$$

$$L^2(a, b) = \left\{ f: [a, b] \rightarrow \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty \right\}$$

In general, for $1 \leq p < \infty$, L^p denotes often the class of functions whose p th power are integrable.

Examples (1) If $f(x)$ is continuous on $[a, b]$ then $f \in L^1(a, b)$, $f \in L^2(a, b)$.

(2) Let $u(x) = \frac{1}{\sqrt{x}}$, $x \in (0, 1]$, $u(0) = 0$.

$$\text{Then } \int_0^1 |u(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_0^1 = 2$$

So, $u \in L^1(0, 1)$.

$$\text{But } \int_0^1 |u(x)|^2 dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{1}{x} dx$$

$$= \lim_{\varepsilon \rightarrow 0^+} \ln x \Big|_{\varepsilon}^1 = +\infty.$$

So, $u \notin L^2(0, 1)$.

Theorem (The Cauchy-Schwarz Inequality)

If $f, g \in L^2(a, b)$, then $fg \in L^1(a, b)$ and

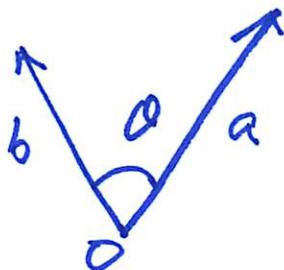
$$\int_a^b |f(x)g(x)| dx \leq \sqrt{\int_a^b |f(x)|^2 dx} \sqrt{\int_a^b |g(x)|^2 dx}.$$

Discrete version: If $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, then

$$|a \cdot b| \leq |a| |b|$$

$$\left| \sum_{k=1}^n a_k b_k \right| \leq \sqrt{\sum_{k=1}^n a_k^2} \sqrt{\sum_{k=1}^n b_k^2}$$

$$n=3: \quad a \cdot b = |a| |b| \cos \theta$$



Proof of Theorem

$$\int_a^b \int_a^b [f(x)g(y) - f(y)g(x)]^2 dx dy \geq 0$$

$$\Rightarrow \int_a^b \int_a^b [f(x)^2 g(y)^2 - 2f(x)f(y)g(x)g(y) + f(y)^2 g(x)^2] dx dy \geq 0$$

$$\Rightarrow \int_a^b \int_a^b f(x)^2 g(y)^2 dx dy + \int_a^b \int_a^b f(y)^2 g(x)^2 dx dy \geq 2 \int_a^b \int_a^b f(x)g(x)f(y)g(y) dx dy$$

$$\Rightarrow 2 \int_a^b f(x)^2 dx \cdot \int_a^b g(y)^2 dy \geq 2 \left(\int_a^b f(x)g(x) dx \right)^2 \quad \underline{\text{Q.E.D.}}$$

Remark. $\|f - g\|_{L^2(a,b)} \stackrel{\text{def.}}{=} \sqrt{\int_a^b |f(x) - g(x)|^2 dx}$ is the Euclidean distance between f and g .

The Thm implies: $\|f - g\|_{L^2} \leq \|f - h\|_{L^2} + \|h - g\|_{L^2}$

(This is the triangle inequality.)

Definition (1) $\{f_n\}$ converges to f in $L^2(a,b)$, denoted $f_n \rightarrow f$ in $L^2(a,b)$, if

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0.$$

(2) $\{f_n\}$ converges to f in $L^1(a,b)$, denoted $f_n \rightarrow f$ in $L^1(a,b)$, if

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)| dx = 0.$$

In general, for $1 \leq p < \infty$, $f_n \rightarrow f$ in $L^p(a,b)$, if

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)|^p dx = 0.$$

Cauchy's criterion: $\{f_n\}$ converges in $L^p(a,b) \iff \lim_{n,m \rightarrow \infty} \int_a^b |f_n - f_m|^p dx = 0$

Examples (1) $f_n(x) = \begin{cases} x^n & x \in [0,1) \\ 1 & x = 1 \end{cases}$. $f(x) = \begin{cases} x^n & x \neq 1 \\ 1 & x = 1 \end{cases}$.
 $f_n \rightarrow f$ on $[0,1)$ pointwise.
 $f_n \rightarrow f$ on $[0,1)$ uniformly. [These have been proved.]

Q: $f_n \rightarrow f$ in $L^2(0,1)$?

$$\int_0^1 |f_n(x) - f(x)|^2 dx = \int_0^1 x^{2n} dx = \frac{1}{2n+1} \rightarrow 0.$$

So, $f_n \rightarrow f$ in $L^2(0,1)$.

A question If we define $g(x) = 0$ for all x in $[0,1)$.

Then we also have $f_n \rightarrow g$ in $L^2(0,1)$.

So the limit of $\{f_n\}$ in $L^2(0,1)$ can be f and g , and $f(x)$ and $g(x)$ differ at $x=1$.

So, two different limits ?

An answer. We need to be precise on

$f = g$ in $L^2(a,b)$. Definition: $f = g$ in $L^2(a,b)$ if $\int_a^b |f(x) - g(x)|^2 dx = 0$.

(2) $f_n(x) = \sin nx$ ($x \in [0, 2\pi]$, $n = 1, 2, 3, \dots$).

Note: $n \gg 1 \Rightarrow \sin nx$ is very oscillatory.

Such functions usually do not converge in L^2 .

Claim $\{\sin nx\}$ does not converge in $L^2(0, 2\pi)$

Proof Otherwise, Cauchy's criterion implies

$$\int_0^{2\pi} |\sin(3nx) - \sin(nx)|^2 dx \rightarrow 0 \text{ (as } n \rightarrow \infty)$$

But, the left-hand side

$$= \int_0^{2\pi} |\sin(3nx)|^2 dx + \int_0^{2\pi} |\sin(nx)|^2 dx - 2 \int_0^{2\pi} \sin(3nx) \cdot \sin(nx) dx$$

$$= 2\pi \quad (n = 1, 2, \dots). \quad \text{A contradiction!}$$

Here, $\int_0^{2\pi} |\sin(kx)|^2 dx = \pi$ ($k = 1, 2, 3, \dots$)

↑
use: $\cos 2\alpha = 1 - 2\sin^2 \alpha$,
 $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

(F)

Some properties

(1) $f_n \rightarrow f$ in $L^2(a, b) \implies \int_a^b |f_n|^2 dx \rightarrow \int_a^b |f|^2 dx$

(2) $f_n \rightarrow f$ in $L^2(a, b) \implies \int_a^b f_n \rightarrow \int_a^b f$ in $L^1(a, b)$

Proof (1) $\left| \int_a^b (|f_n|^2 - |f|^2) dx \right| = \left| \int_a^b (f_n - f)(f_n + f) dx \right|$

$$= \left| \int_a^b (f_n - f)(f_n - f + 2f) dx \right|$$

$$= \left| \underbrace{\int_a^b (f_n - f)^2 dx}_{\rightarrow 0} + \int_a^b 2f(f_n - f) dx \right|$$

$\rightarrow 0$
as $n \rightarrow \infty$

$$\left| \int_a^b f(f_n - f) dx \right| \leq \int_a^b |f| |f_n - f| dx$$

$$\leq \sqrt{\int_a^b |f|^2 dx} \cdot \sqrt{\int_a^b |f_n - f|^2 dx} \rightarrow 0.$$

$$(2) \int_a^b |f_n - f| dx = \int_a^b 1 \cdot |f_n - f| dx$$

$$\leq \sqrt{\int_a^b 1^2 dx} \cdot \sqrt{\int_a^b |f_n - f|^2 dx} \rightarrow 0. \quad \underline{\text{Q.E.D.}}$$

Definition Let $f, f_n \in L^2(a, b)$ ($n = 1, 2, \dots$). $\{f_n\}$ converges to f weakly in $L^2(a, b)$, denoted $f_n \rightarrow f$ weakly in $L^2(a, b)$ if

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) g(x) dx = \int_a^b f(x) g(x) dx \quad \text{for any } g \in L^2(a, b).$$

Often call $f_n \rightarrow f$ in $L^2(a, b)$, strong convergence in $L^2(a, b)$.

Example $\{\sin nx\}$ converges weakly to 0 in $L^2(0, 2\pi)$. i.e.,

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} \sin(nx) g(x) dx = 0$$

for any $g(x)$ in $L^2(a, b)$.
True also for $\{\cos nx\}$.

These are: The Riemann-Lebesgue Lemma.

Proposition $f_n \rightarrow f$ weakly in $L^2(a, b)$
 $\implies f_n \rightarrow f$ on $[a, b]$ in average. i.e.,

$$\lim_{n \rightarrow \infty} \int_c^d f_n(x) dx = \int_c^d f(x) dx$$
for any subinterval $[c, d] \subseteq [a, b]$.

Proof Choose $g(x) = \begin{cases} 1 & \text{if } x \in [c, d] \\ 0 & \text{if } x \notin [c, d] \end{cases}$. Q.E.D.

Theorem $f_n \rightarrow f$ weakly in $L^2(a, b) \iff$ The sequence $\left\{ \int_a^b |f_n|^2 dx \right\}$ is bounded and $f_n \rightarrow f$ on $[a, b]$ in average.

Theorem Let $f, f_n \in L^2(a, b)$ ($n = 1, 2, \dots$).

$f_n \rightarrow f$ unif. on $[a, b] \implies f_n \rightarrow f$ in $L^2(a, b)$
 $\Downarrow \nleftrightarrow$
 $f_n \rightarrow f$ pointwise on $[a, b] \not\iff f_n \rightarrow f$ weakly in $L^2(a, b)$
 $\Downarrow \nleftrightarrow$

Proof $\int_a^b |f_n(x) - f(x)|^2 dx \leq \left(\sup_{x \in [a, b]} |f_n(x) - f(x)| \right)^2 (b-a)$

$$\left| \int_a^b (f_n - f) dx \right| \leq \sqrt{\int_a^b (f_n - f)^2 dx} \cdot \sqrt{\int_a^b 1^2 dx}$$

$\{ \sin nx \}$. $\{ x^n \}$. Q.E.D.

Theorem $f_n \rightarrow f$ in $L^2(a, b) \iff f_n \rightarrow f$ weakly in $L^2(a, b)$ and $\|f_n\|_{L^2(a, b)} \rightarrow \|f\|_{L^2(a, b)}$. i.e.,

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x)|^2 dx = \int_a^b |f(x)|^2 dx.$$

Proof Only " \Leftarrow " as " \Rightarrow " has been proved.

$$\begin{aligned} & \int_a^b |f_n(x) - f(x)|^2 dx \\ &= \int_a^b |f_n(x)|^2 dx + \int_a^b |f(x)|^2 dx - 2 \int_a^b f_n(x) f(x) dx \\ &\longrightarrow \int_a^b |f(x)|^2 dx + \int_a^b |f(x)|^2 dx - 2 \int_a^b f(x) \cdot f(x) dx \\ &= 0. \end{aligned}$$

Q.E.D.