



Chapter 1 Sequences and Series

§1.4 Infinite series of functions. Power series

We study infinite series of functions $f_n = f_n(x)$ defined on an interval I of \mathbb{R} : $\sum_{n=1}^{\infty} f_n(x)$.

Two main classes of such series.

① power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$.

② Fourier series

$$\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\text{or } \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Definition ① $S_n(x) = \sum_{k=1}^n f_k(x)$, ($n = 1, 2, \dots$)

These are partial sums. $\{S_n(x)\}$ is a seq. of functions on I .

② $\sum_{n=1}^{\infty} f_n(x) = f(x)$ means $\lim_{n \rightarrow \infty} S_n(x) = f(x)$ $(x \in I)$

In this case, call $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f(x)$ on I .

③ If $S_n(x) \rightarrow f(x)$ uniformly on I , then we say $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$, uniformly on I .

[2]

Theorem Let $I = [a, b]$.

(1) Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges unif. to $f(x)$ on I and for $x_0 \in I$, $\lim_{x \rightarrow x_0} f_n(x) = A_n$ exists. Then

$$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} f_n(x).$$

(2) Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges unif. to $f(x)$ on I and each f_n is integrable on $I = [a, b]$ ($n=1, 2, \dots$). Then $f(x)$ is also integrable on $I = [a, b]$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x) dx.$$

(3) Suppose $\sum_{n=1}^{\infty} f_n'(x)$ converges unif. to $f'(x)$ on $[a, b]$. and for some $x_0 \in [a, b]$, $\sum f_n(x_0)$ converges to $F(x_0)$. Then $\sum f_n(x)$ converges to $f(x)$ unif. on $[a, b]$, and

$$\frac{d}{dx} \left(\sum_{n=1}^{\infty} f_n(x) \right) = \sum_{n=1}^{\infty} \frac{df_n(x)}{dx} \quad (x \in [a, b]).$$

Theorem (Weierstrass Test) Suppose $M_n \geq 0$ ($n=1, 2, \dots$)

and ① $|f_n(x)| \leq M_n$ ($n=1, 2, \dots$; $x \in I$);

② $\sum_{n=1}^{\infty} M_n$ converges.

Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I .

Proof Use Cauchy's criterion:

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| \leq \sum_{k=n+1}^{n+p} M_k \rightarrow 0 \text{ (as } n \rightarrow \infty\text{).}$$

Q.E.D.

Examples (1) $f_n(x) = \frac{1}{1+n^2x^2}$, $x \in [0, 1]$.

$x=0 \Rightarrow f_n(0)=1 \Rightarrow \sum_{n=1}^{\infty} f_n(0)$ diverges.

$0 < x \leq 1 \Rightarrow \sum_{n=1}^{\infty} f_n(x)$ converges. since for a

fixed $x > 0$. $\lim_{n \rightarrow \infty} \frac{\frac{1}{1+n^2x^2}}{\frac{1}{n^2}} = \frac{1}{x^2} > 0$.

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

So, $\sum_{n=1}^{\infty} f_n(x)$ converges on $(0, 1]$ pointwise.

Claim $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly on $(0, 1]$.

Pf $S_n(x) = f_1(x) + f_2(x) + \dots + f_n(x)$ for $x = \frac{1}{n}$

$$\left| S_{n+1}(x) - S_n(x) \right| = |f_{n+1}(x)|$$

$$\sup_{0 < x \leq 1} |S_{n+1}(x) - S_n(x)| = \sup_{0 < x \leq 1} f_{n+1}(x) \geq \frac{1}{1+n^2(\frac{1}{n})^2} = \frac{1}{2}$$

$\not\rightarrow 0$ as $n \rightarrow \infty$. Q.F.D.

Claim $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in $[\alpha, 1]$ for any $\alpha \in (0, 1)$.

Pf $|f_n(x)| = \underbrace{\frac{n^2 x^2}{1+n^2 x^2}}_{\leq 1} \cdot \frac{1}{n^2 x^2} \leq \underbrace{\frac{1}{n^2}}_{M_n} \cdot \frac{1}{\alpha^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{\alpha^2} = \frac{1}{\alpha^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\sum_{n=1}^{\infty} f_n(x)$ converges unif. on $[\alpha, 1]$. Q.F.D.

Power Series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-3)^n.$$

Consider $\sum_{n=0}^{\infty} a_n x^n$. a_n : real numbers.

Observe



(1) Suppose

$$\sum_{n=1}^{\infty} a_n x_1^n \text{ converges for some } x_1 > 0$$

Then

$$\sum_{n=1}^{\infty} a_n x^n \text{ converges absolutely for } |x| < x_1$$

(2) Suppose

$$\sum_{n=1}^{\infty} a_n x_2^n \text{ diverges for some } x_2 > 0$$

then $\sum_{n=1}^{\infty} a_n x^n$ diverges for all $|x| > x_2$.

So, there exists a threshold R — called
radius of convergence

Justify: (1) $|a_n x^n| = |a_n x_1^n| \cdot \left|\frac{x}{x_1}\right|^n \leq A \left|\frac{x}{x_1}\right|^n$

$$\Rightarrow \left|\frac{x}{x_1}\right| < 1. \quad \sum a_n x_1^n \text{ converges} \Rightarrow a_n x_1^n \rightarrow 0$$

$$\Rightarrow |a_n x_1^n| \leq A \quad (n=1, 2, \dots) \text{ for some } A > 0.$$

Now, $\sum_{n=0}^{\infty} A \left|\frac{x}{x_1}\right|^n$ converges. So, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

Now, $\sum a_n x^n$ unif. converges in $[0, x_1 - \varepsilon]$ for $\varepsilon > 0$ small.

(2) Similarly.

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Theorem For any (complex) power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

there exists a unique radius of convergence $R \geq 0$

such that

$|z - z_0| < R$: it converges absolutely

$[|z - z_0| < R - \varepsilon$: it converges uniformly.]

$|z - z_0| > R$: it diverges.

Examples (1) $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ ($x \in \mathbb{R}$)

$$a_n = \left| \frac{1}{n} x^n \right|, a_{n+1} = \frac{1}{n+1} |x^{n+1}|$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} |x|^{n+1}}{\frac{1}{n} |x|^n} = |x|$$

So, $|x| < 1 \Rightarrow$ convergence.

$|x| > 1 \Rightarrow$ divergence.

$$\boxed{R = 1}$$

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergence.

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergence.

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$. Similar: $\boxed{R = 1}$

$x = 1$: $\sum \frac{1}{n^2}$ convergence

$x = -1$: $\sum \frac{(-1)^n}{n^2}$ convergence.

(2) $\sum_{n=0}^{\infty} \frac{1}{n 2^n} (x - 3)^n$.

$$a_n = \frac{1}{n 2^n} |x - 3|^n, \quad \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{|x - 3|}{2} \Leftrightarrow$$

$\frac{|x-3|}{2} < 1$, i.e. $|x-3| < 2 \Rightarrow$ convergence

$\frac{|x-3|}{2} > 1$ i.e. $|x-3| > 2 \Rightarrow$ divergence.

So, $\boxed{R = 2}$.

Theorem Suppose the radius of convergence of $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ is $R > 0$. Let $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$

for $|x-x_0| < R$.

① f is infinitely differentiable in $(x_0, |x-x_0| < R)$,

$$f'(x) = \sum_{k=1}^{\infty} a_k k (x-x_0)^{k-1}$$

$$f''(x) = \sum_{k=2}^{\infty} a_k k(k-1) (x-x_0)^{k-2}$$

.....

All these derivative power series have the same radii of convergence R .

In particular

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad (n = 0, 1, 2, \dots).$$

② For any $[c, d] \subset (x_0 - R, x_0 + R)$.

$$\int_a^b f(x) dx = \sum_{k=0}^{\infty} a_k \int_a^b (x-x_0)^k dx.$$

Application to calculating infinite series.

Example $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = ?$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = ?$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \underbrace{\sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^2} &= \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx \quad R=1 \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \end{aligned}$$

$$\text{But the left-hand side} = \arctan x \Big|_0^1 = \frac{\pi}{4}$$

$$\text{So, } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.$$

Yes, but not rigorous! A bit more careful:

$$\int_0^{1-\varepsilon} \frac{dx}{1+x^2} = \arctan x \Big|_0^{1-\varepsilon} = \arctan(1-\varepsilon) \rightarrow$$

$$\arctan 1 = \frac{\pi}{4} \quad (\text{as } \varepsilon \rightarrow 0^+)$$

$$\int_0^{1-\varepsilon} \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{1-\varepsilon} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (1-\varepsilon)^{2n+1} \xrightarrow{\text{legal}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \quad (\text{as } \varepsilon \rightarrow 0^+)$$

True, because of

Theorem If $\sum_{n=0}^{\infty} c_n$ converges and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ ($-1 < x < 1$)
then $\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} c_n$.