

Chapter 2 Matrix Techniques

Section 2.1 Determinant and Rank

Section 2.2 Eigenvalues and Eigenvectors

Section 2.3 Two classes of matrices:

Symmetric Positive Definite (SPD) matrices; ~~Rot.~~ Orthogonal (and Rotational) matrices.

Section 2.4 Matrix Exponentials.

Section 2.1 Determinant and Rank

Definition of determinant of a (square) matrix.

only for a square matrix

$A = [a]$ a is a number. A : 1×1 matrix

$\det A = a$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a, b, c, d : numbers.

$\det A = ad - bc$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det A = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Now, consider an $n \times n$ matrix

$$A = [a_{ij}] = [a_{ij}]_{n \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let $P_n = \{\text{all permutations of } (1 2 \cdots n)\}$

A permutation is a rearrangement, or is a bijection (1-1, onto mapping).

P_n has $n!$ elements. e.g., $n=3$. P_3 consists of $(1\ 2\ 3)$, $(1\ 3\ 2)$, $(2\ 1\ 3)$, $(2\ 3\ 1)$, $(3\ 1\ 2)$, $(3\ 2\ 1)$.
 $6 = 3!$ permutations of $(1\ 2\ 3)$. Note that $(1\ 2\ 3)$ is also a permutation of $(1\ 2\ 3)$.

An element / permutation of P_n is often denoted
 $(j_1\ j_2\ \cdots\ j_n)$

If $(j_1\ j_2\ \cdots\ j_n) \in P_n$, we denote

$$\sigma(j_1\ j_2\ \cdots\ j_n) = \text{sign } \prod_{1 \leq p < q \leq n} (j_q - j_p)$$

This is either 1 or -1.

If $\sigma(j_1\ j_2\ \cdots\ j_n) = 1$ then $(j_1\ j_2\ \cdots\ j_n)$ is an even permutation.

If $\sigma(j_1\ j_2\ \cdots\ j_n) = -1$ then $(j_1\ j_2\ \cdots\ j_n)$ is an odd permutation.

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These terminologies are related to the fact that each $(j_1 j_2 \dots j_n) \in P_n$ is obtained by finitely many pair inter exchanges. The smallest number of such pair exchanges is even $\xrightarrow[\text{or}]{\text{odd}} s(j_1 j_2 \dots j_n) = 1$ (or 2).

Example $n=3$. $(123) \in P_3$. $s(123) = \text{sign } (-1)(3-1)(3-2) = 1$. So, (123) is an even permutation. We need 0 steps of pair ~~exchanges~~ to obtain (123) from (123) .

$(231) \in P_3$: $s(231) = \text{sign } (3-2)(1-2)(1-3) = 1$

even. $\begin{matrix} 1 \leftrightarrow 2 & 1 \leftrightarrow 3 \\ (123) \rightarrow (213) \rightarrow (231). \end{matrix}$

m.m. steps of pair ~~exchanges~~ = 2. even.

$(132) \in P_3$ $(123) \rightarrow (132)$ odd.

Definition. Let $A = [a_{ij}]_{n \times n}$.

$$\det A = \sum_{(j_1 j_2 \dots j_n) \in P_n} s(j_1 j_2 \dots j_n) a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

Examples (i) $n=2$. $P_2 = \{(12), (21)\}$.

(12) : $a_{11} a_{22}$. $s(12) = 1$.

(21) : $a_{12} a_{21}$. $s(21) = -1$.

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$

(2) $n=3$, $P_3 = \{(123), (132), (213), (231), (312), (321)\}$.

(123) . $\text{sign}(123) = 1$. $a_{11} a_{22} a_{33}$

(132) $\text{sign}(132) = -1$ $-a_{11} a_{23} a_{32}$

(213) $\text{sign}(213) = -\text{sign}(123) = -1$. $-a_{12} a_{21} a_{33}$

(312) ($\rightarrow (213) \rightarrow (123)$) $\text{sign}(312) = 1$. $a_{13} a_{21} a_{32}$

(321) $\text{sign}(321) = 1$ $a_{12} a_{23} a_{31}$

So, $\det A = \det [a_{ij}]_{3 \times 3}$

$$= a_{11} a_{22} a_{33} + a_{13} a_{21} a_{32} + a_{12} a_{23} a_{31} \\ - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}.$$

Properties of determinants

Theorem Let $A = [a_{ij}]_{n \times n}$. Then,

$$(1) \det A = \sum_{(j_1 j_2 \dots j_n) \in P_n} \text{sign}(j_1 j_2 \dots j_n) a_{j_1 1} a_{j_2 2} \dots a_{j_n n}.$$

$$(2) \det A^T = \det A.$$

(1) \uparrow
the transpose of A . e.g. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem (2) Two rows (or columns) are interchanged
 \Rightarrow the sign of determinant is reversed.

e.g. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

In particular, two rows (or columns) are identical. Then the $\det = 0$.

In particular, one row = 0 $\Rightarrow \det = 0.$

(3) If a row (column) is multiplied by a number c , then the det. is multiplied by $c.$

e.g., $\det \begin{bmatrix} 1 & 2 \\ 4 & 12 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 4 \cdot 1 & 4 \cdot 3 \end{bmatrix} = 4 \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = 4 \cdot 1 = 4.$

(4) If a multiple of a row (column) is subtracted from another row (column), then the det. is unchanged.

e.g., $\det \begin{bmatrix} 1 & 2 \\ 4 & 12 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = 1 \cdot 4 - 0 \cdot 2 = 4.$

In particular, two rows are parallel (or same) $\Rightarrow \det A = 0.$

Theorem (Row or Column expansion) Let $A = [a_{ij}]_{n \times n}.$

Then for each i ($1 \leq i \leq n$)

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

(row expansion)
for each j ($1 \leq j \leq n$)

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}.$$

Here, $C_{ij} = (-1)^{i+j} \det A_{ij}$

A_{ij} is the ~~$n-i$~~ $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j in $A.$

Call C_{ij} the (i,j) -th cofactor.

Also for $i \neq k:$ $\sum_{j=1}^n a_{ij} C_{kj} = 0, \sum_{j=1}^n a_{ji} C_{jk} = 0.$

Example

$$\det \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\stackrel{i=1}{=} 1 \cdot (-1)^{1+1} \det \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} + 0 \cdot (-1)^{1+2} \det \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

$$+ (-2) \cdot (-1)^{1+3} \det \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= 1 \cdot (-1 - 2) - 2 \cdot 3 = -9.$$

Theorem

$$\det \begin{bmatrix} a_{11} & & 0 \\ a_{21} & \ddots & \\ 0 & \ddots & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

↑
diagonal matrix

$$\det I = \det \begin{bmatrix} 1 & 0 \\ 0 & \ddots \end{bmatrix} = 1.$$

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & \ddots & & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

upper triangular matrix.

$$\det \begin{bmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

lower triangular matrix

Theorem "Let A and B be two $n \times n$ matrices.

$$\text{Then } \det(AB) = (\det A)(\det B)$$

(2) If $A = [a_{ij}]_{n \times n}$ is invertible, i.e.,

$$AB = BA = I$$

for some $n \times n$ matrix B , ~~($B = A^{-1}$; inverse)~~, then
 $\det A^{-1} = (\det A)^{-1}$.

Moreover

$$A^{-1} = \frac{1}{\det A} C^T$$

where $C = [C_{ij}]$ is the co-factor matrix of A .

Theorem (Cramer's Rule) Assume $A = [a_{ij}]_{n \times n}$
 is invertible $b = [b_1 \ b_2 \ \dots \ b_n]^T$, $x = [x_1 \ x_2 \ \dots \ x_n]^T$, and

$$Ax = b.$$

Then the unique solution x is given by

$$x_j = \frac{\det A_j}{\det A} \quad (j=1, 2, \dots, n)$$

where A_j is the $n \times n$ matrix obtained by
 replacing the j th column of A by b .

Derivative of determinant

Let $A(t) = [a_{ij}(t)]_{n \times n}$. Each $a_{ij}(t)$ is a (differentiable) function of t in an interval I .

$$A(t) = \begin{bmatrix} a_{11}(t) \\ a_{12}(t) \\ \vdots \\ a_{1n}(t) \end{bmatrix} \quad \text{each row } a_i(t) = (a_{i1}(t), a_{i2}(t), \dots, a_{in}(t))$$

$$= \begin{bmatrix} b_1(t) & b_2(t) & \dots & b_n(t) \end{bmatrix} \quad \text{each column } b_j(t) = \begin{bmatrix} b_{1j}(t) \\ b_{2j}(t) \\ \vdots \\ b_{nj}(t) \end{bmatrix}$$

Theorem $\frac{d}{dt} \det A(t) = \sum_{i=1}^n \det \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1i-1}(t) & a_{1i+1}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2i-1}(t) & a_{2i+1}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{ni-1}(t) & a_{ni+1}(t) & \dots & a_{nn}(t) \end{bmatrix}$

$$= \sum_{j=1}^n \det [b_1(t) \dots b_{j-1}(t) b'_j(t) b_{j+1}(t) \dots \dots b_n(t)].$$

Proof The case $n=2$

$$\begin{aligned} & \frac{d}{dt} \det \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \\ &= \frac{d}{dt} [a(t)d(t) - b(t)c(t)] \\ &= a'(t)d(t) + a(t)d'(t) - b'(t)c(t) - b(t)c'(t) \\ &= [a'(t)d(t) - \cancel{a(t)b'(t)c(t)}] + [a(t)d'(t) - b(t)c'(t)] \\ &= \det \begin{bmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{bmatrix} + \det \begin{bmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{bmatrix}. \end{aligned}$$

The general case: use the definition of det.

$$\det A(t) = \sum_{(j_1 j_2 \cdots j_n) \in P_n} \text{sign}(j_1 j_2 \cdots j_n) a_{1j_1}(t) a_{2j_2}(t) \cdots a_{nj_n}(t).$$

$$\begin{aligned} \frac{d}{dt} (\det A(t)) &= \sum_{(j_1 j_2 \cdots j_n) \in P_n} \text{sign}(j_1 j_2 \cdots j_n) \frac{d}{dt} (a_{1j_1}(t) a_{2j_2}(t) \cdots a_{nj_n}(t)) \\ &\quad \frac{d}{dt} (a_{1j_1}(t) a_{2j_2}(t) \cdots a_{nj_n}(t)) \\ &= a_{1j_1}'(t) a_{2j_2}(t) \cdots a_{nj_n}(t) + a_{1j_1}(t) a_{2j_2}'(t) a_{3j_3}(t) \cdots a_{nj_n}(t) \\ &\quad + \cdots + a_{1j_1}(t) \cdots a_{n-1j_{n-1}}(t) a_{nj_n}'(t). \quad \underline{\text{Q.E.D.}} \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dt} \begin{vmatrix} t & e^{-t} & 1 \\ t^2 & 1 & e^t \\ 0 & \sin t & \cos t \end{vmatrix} &= \begin{vmatrix} 1 & e^{-t} & 1 \\ 2t & 1 & e^t \\ 0 & \sin t & \cos t \end{vmatrix} \\ &+ \begin{vmatrix} t & -e^{-t} & 1 \\ t^2 & 0 & e^t \\ 0 & \cos t & \cos t \end{vmatrix} + \begin{vmatrix} t & e^{-t} & 0 \\ t^2 & 1 & e^t \\ 0 & \sin t & -\cos t \end{vmatrix} \\ &= \cos t + 2t \sin t - e^t \sin t + 2t e^{-t} \cos t \\ &\quad + t^2 \cos t - t e^t \cos t + t^2 e^{-t} \cos t \\ &\quad - t \cos t - t e^t \sin t + t^2 e^{-t} \cos t \\ &= \cancel{\cos t} - (1-t+t^2) \cos t + 2t \sin t \\ &\quad - t e^t \cos t - (1+t) e^t \sin t + (2t^2-2t) e^{-t} \cos t. \end{aligned}$$

Example (Wronskians). Let $y = y_1(t)$ and $y = y_2(t)$ be two solutions to

$$y'' + p(t)y' + q(t)y = 0$$

for t in an interval I , where $p(t), q(t)$ are continuous and bounded functions on I .

Call $W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$

a Wronskian. (or the Wronskian associated with $y_1(t), y_2(t)$).

$$\begin{aligned} W'(t) &= \underbrace{\begin{vmatrix} y_1'(t) & y_2'(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}_{=0} + \begin{vmatrix} y_1(t) & y_2(t) \\ y_1''(t) & y_2''(t) \end{vmatrix} \\ &= y_1(t)y_2''(t) - y_2(t)y_1''(t) \end{aligned}$$

Since $y_1(t), y_2(t)$ are solns.

$$y_i''(t) = -p(t)y_i' - q(t)y_i, \quad i=1, 2.$$

$$\begin{aligned} \text{So, } W'(t) &= y_1(-p(t)y_2' - q(t)y_2) \\ &\quad - y_2(-p(t)y_1' - q(t)y_1) \\ &= -p(t)[y_1'y_2 - y_2'y_1] = -p(t)\underbrace{(y_1'y_2 - y_2'y_1)}_{=W(t)} \\ &= -p(t)W(t). \end{aligned}$$

$$\boxed{W' = -p(t)W}$$

$$W(t) = C e^{-\int p(t)dt}$$

$$\text{So, } W(t) = 0 \quad (\text{in case } C=0)$$

$$\text{or } W(t) \neq 0 \quad \forall t \quad (\text{in case } C \neq 0).$$