Math 210. Lecture Notes. Bo Li, Fall 2013

Chapter 2 Matrix Techniques

Section 2.1 Determinant and Rank
Section 2.2 Eigenvalues and Eigenvectors
Section 2.3 Two classes of matrices: Symmetric Positive Definite (SPD) matrices; ~~Rot~~. Orthogonal (and Rotational) matrices.
Section 2.4 Matrix Exponentials.

Section 2.1 Determinant and Rank

Definition of $\underbrace{\text{determinant}}_{\text{only for a }\underline{\text{square}}\text{ matrix}}$ of a (square) matrix.

$A = [a]$ a is a number. A: 1×1 matrix
$\det A = a$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a, b, c, d: numbers.
$\det A = ad - bc$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det A = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Now, consider an $n \times n$ matrix

$$A = [a_{ij}] = [a_{ij}]_{n \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

<u>Let</u> $P_n = \{$ all permutations of $(1\ 2 \cdots n)\}$

A permutation is a rearrangement, or is a bijection (1-1, onto mapping).

P_n has $n!$ elements. e.g., $n=3$. P_3 consists of $(1\ 2\ 3)$, $(1\ 3\ 2)$, $(2\ 1\ 3)$, $(2\ 3\ 1)$ $(3\ 1\ 2)$ $(3\ 2\ 1)$

$6 = 3!$ permutations of $(1\ 2\ 3)$. <u>Note that</u> $(1\ 2\ 3)$ is also a permutation of $(1\ 2\ 3)$.

An element / permutation of P_n is often denoted

$$(j_1\ j_2 \cdots j_n)$$

If $(j_1\ j_2 \cdots j_n) \in P_n$, we <u>denote</u>

$$s(j_1 j_2 \cdots j_n) = \text{sign} \prod_{1 \le p < q \le n} (j_q - j_p)$$

This is either 1 or -1.

If $s(j_1 j_2 \cdots j_n) = 1$ then $(j_1\ j_2 \cdots j_n)$ is an <u>even permutation</u>.

If $s(j_1 j_2 \cdots j_n) = -1$ then $(j_1 j_2 \cdots j_n)$ is an <u>odd permutation</u>.

These terminologies are related to the fact that each $(j_1 j_2 \cdots j_n) \in P_n$ is obtained by finitely many pair ~~exchanges~~ interchanges. The smallest number of such pair ~~exchanges~~ interchanges is even (or odd) $\Longleftrightarrow s(j_1 j_2 \cdots j_n) = 1$ (or -1).

<u>Example</u> $n=3$. $(123) \in P_3$. $s(123) = \text{sign}\,(2-1)(3-1)(3-2)$ $=1$. So, (123) is an even permutation. We need 0 steps of pair ~~exchanges~~ interchanges to obtain (123) from (123).

$(231) \in P_3$: $s(231) = \text{sign}\,(3-2)(1-2)(1-3) = 1$

even.

$$(123) \xrightarrow{1 \leftrightarrow 2} (213) \xrightarrow{1 \leftrightarrow 3} (231).$$

min. steps of pair ~~exchanges~~ interchanges $= 2$. even.

$(132) \in P_3$ $\quad (123) \longrightarrow (132)$ odd.

<u>Definition</u>. Let $A = [a_{ij}]_{n \times n}$.

$$\det A = \sum_{(j_1 j_2 \cdots j_n) \in P_n} s(j_1 j_2 \cdots j_n)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

<u>Examples</u> (1) $n=2$. $P_2 = \{(12), (21)\}$.

(12): $a_{11} a_{22}$. $\quad s(12) = 1$.

(21): $a_{12} a_{21}$ $\quad s(21) = -1$.

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$

(2) $n=3$, $P_3 = \{(123), (132), (213), (231), (312)(321)\}$.

(123). $\operatorname{sign}(123)=1$. $a_{11}a_{22}a_{33}$

(132) $\operatorname{sign}(132)=-1$ $-a_{11}a_{23}a_{32}$

(213) $\operatorname{sign}(213) = -\operatorname{sign}(123) = -1$. $-a_{12}a_{21}a_{33}$

(312) $(\to(213)\to(123))$ $\operatorname{sign}(312)=1$. $a_{13}a_{21}a_{32}$

(312) $\operatorname{sign}(312)=1$ $a_{13}a_{21}a_{32}$

(321) $\operatorname{sign}(321)=-1$ $-a_{13}a_{22}a_{31}$

So, $\det A = \det [a_{ij}]_{3\times 3}$

$$= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Properties of determinants

Theorem Let $A = [a_{ij}]_{n\times n}$. Then.

(1) $$\det A = \sum_{(j_1 j_2 \cdots j_n)\in P_n} \operatorname{sign}(j_1 j_2 \cdots j_n)\, a_{j_1 1} a_{j_2 2} \cdots a_{j_n n}.$$

(2) $\det A^T = \det A$.

the transpose of A. e.g. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem (1) Two rows (or columns) are interchanged $\Rightarrow$ the sign of determinant is reversed.

e.g. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

In particular, two rows (or columns) are identical, then the $\det = 0$.

(2) If a row (column) is multiplied by a number c, then the det. is multiplied by c.

e.g., $\det\begin{bmatrix} 1 & 2 \\ 4 & 12 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 \\ 4\cdot 1 & 4\cdot 3 \end{bmatrix} = 4\det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
$= 4\cdot 1 = 4$.

(3) If a multiple of a row (column) is subtracted from another row (column), then the det. is unchanged.

e.g., $\det\begin{bmatrix} 1 & 2 \\ 4 & 12 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = 1\cdot 4 - 0\cdot 2 = 4$.

<u>Theorem</u> (Row or Column expansion) Let $A = [a_{ij}]_{n\times n}$.
Then for each i $(1 \le i \le n)$

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$
(row expansion)

for each j $(1 \le j \le n)$

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{jn} C_{jn}.$$

Here, $C_{ij} = (-1)^{i+j} \det A_{ij}$.

A_{ij} is the ~~m~~ $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j in A.

Call C_{ij} the (i,j)-th <u>cofactor</u>.

<u>Example</u>

$$\det\begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\overset{i=1}{=} 1\cdot(-1)^{1+1}\det\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} + 0\cdot(-1)^{1+2}\det\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

$$+ (-2)\cdot(-1)^{1+3}\det\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= 1\cdot(-1-2) - 2\cdot 3 = -\cancel{8}9.$$

<u>Theorem</u>

$$\det\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{bmatrix} = a_{11}\, a_{22} \cdot \cdots \cdot a_{nn}$$

diagonal matrix

$$\det I = \det\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = 1.$$

$$\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} = a_{11}\, a_{22} \cdots a_{nn}$$

upper triangular matrix.

$$\det\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11}\, a_{22} \cdots\cdots a_{nn}$$

lower triangular matrix

<u>Theorem</u> (1) Let A and B be two $n \times n$ matrices.
Then $\det(AB) = (\det A)(\det B)$

(2) If $A = [a_{ij}]_{n\times n}$ is intertible. i.e.

$$AB = BA = I$$

for some $n \times n$ matrix B. ~~then~~ ($B = A^{-1}$: inverse), then

$$\det A^{-1} = (\det A)^{-1}.$$

Moreover

$$A^{-1} = \frac{1}{\det A} C^T$$

where $C = [C_{ij}]$ is the co-factor matrix of A.

<u>Theorem</u> (Cramer's Rule) Assume $A = [a_{ij}]_{n\times n}$ is invertible $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. and

$$Ax = b.$$

Then the unique solution x is given by

$$x_j = \frac{\det A_j}{\det A} \quad (j = 1, 2, \cdots, n)$$

where A_j is the $n \times n$ matrix obtained by replacing the jth column of A by b.