

Matrix Rank (It's a nonnegative integer.)

II

Begin with rank of a group of vectors.

Notation $\mathbb{R}^n = \left\{ \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} : a_{ij} \in \mathbb{R} \right\}$. Similar for \mathbb{C}^n .

Recall! $\vec{u}_1, \dots, \vec{u}_k$: linearly dependent if there exist c_1, \dots, c_k real numbers, not all 0, such that $c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_k \vec{u}_k = \vec{0}$

Otherwise, called linearly independent.

This is $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (n zeros).

meaning: $c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_k \vec{u}_k = \vec{0} \implies$ all $c_1 = c_2 = \dots = c_k = 0$.

A very elementary but useful fact:

Theorem $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ are linearly dependent

\iff one of them is a linear combination of the others.

e.g., $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

the right-side is a linear combination of the 3 vectors

So, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

Note: $\{\vec{0}\}$ is linearly dependent.

Any group of vectors containing $\vec{0}$ is linearly dependent.

Given a group of vectors in \mathbb{R}^n :

$$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k.$$

If all of them are $\vec{0}$, then we define the rank of these vectors to be 0.

Otherwise, pick up the 1st nonzero vector in the list, say, $\vec{u}_1 \neq \vec{0}$.

If \vec{u}_1, \vec{u}_2 are linearly indep. then keep \vec{u}_1, \vec{u}_2 .
Otherwise only keep \vec{u}_1 . Denote by S the set of vectors that we keep.

If \vec{u}_3 is a linear combination of vectors in S , then "throw it away" — do not keep it in S . Otherwise, keep it in S .
Keepy point: S is linearly indep.

Finally we find a subgroup S of vectors:

- ① S is linearly indep.
- ② Any vector (thrown away or not) is a linear combination of vectors in S .

or: a max. linearly indep. subgroup

Call S a maximal subgroup of $\vec{u}_1, \dots, \vec{u}_k$.

Maximal? yes! If you add one more vector to S , then it becomes linearly dependent.

Example $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$S: \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

Note $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a linear combination of vectors in S . So, S is maximal.

A different max. subgroup is: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Theorem Given a group of finitely many vectors in \mathbb{R}^n . Any two max. subgroups of these vectors have the same number of vectors.

Definition. Given vectors $\vec{u}_1, \dots, \vec{u}_k$ in \mathbb{R}^n .

$\text{rank}(\vec{u}_1, \dots, \vec{u}_k) \leftarrow$ the rank of this group of vectors
 $= \begin{cases} 0 & \text{if all } \vec{u}_1 = \vec{0}, \dots, \vec{u}_k = \vec{0} \\ \# \text{ vectors in any } \underline{\text{max.}} \text{ subgroup.} \end{cases}$

Now, matrix rank.

Write a matrix as

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix} = [\vec{b}_1 \vec{b}_2 \cdots \vec{b}_n]$$

each $\vec{a}_i \in \mathbb{R}^n$, each $\vec{b}_j \in \mathbb{R}^m$.

\uparrow ~~row vectors~~ $\vec{a}_i = [a_{i1} \ a_{i2} \ \cdots \ a_{in}]^T$

$\vec{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}$; ~~column vectors~~.

Terminology Row rank of a matrix is the rank of row vectors of the matrix.
Column rank ... (similar).

A magical result

Theorem For any matrix. row rank = col. rank

Think about this: Randomly generate 100 numbers to form a 10×10 matrix. Then, row rank = col. rank. In particular, if all rows are linearly independent, then all columns are linearly indep.!

Proof of Thm show row rank \leq col. rank.
col rank \leq row rank.

To see the idea, consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Suppose row rank = 2. So, there are two rows that are linearly independent. Assume they are the first two rows. So

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only trivial solution $x_1 = 0, x_2 = 0$.

This system is
$$\begin{cases} a_{11}x_1 + a_{21}x_2 = 0 \\ a_{12}x_1 + a_{22}x_2 = 0 \\ a_{13}x_1 + a_{23}x_2 = 0 \end{cases}$$

The max. number of linearly independent equations. i.e. vectors (a_{11}, a_{21}) , (a_{12}, a_{22}) and (a_{13}, a_{23}) must be ≥ 2 . For otherwise, only one equation is indep. and the other two are dependent. So, solns become those of one equation, e.g. $a_{11}x_1 + a_{21}x_2 = 0$.

unknowns \geq # eq. So, there are infinitely many solutions. Contradiction!

Therefore $\text{rank} \left(\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \right) \geq 2$

Finally $\implies \text{rank} \left(\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right) = \text{col. rank} \geq 2$

Similar for col. rank \leq row rank.

Q.E.D.

Theorem Let A be an $n \times n$ (nonzero) matrix.

Assume (1) A has a $r \times r$ nonzero subdeterminant.
 (2) All $(r+1) \times (r+1)$ subdeterminants of A vanish.

Then $\text{rank}(A) = r$.

Example ① $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 1 \neq 0$

Any 3×3 subdeterminant = 0.

$\Rightarrow \text{rank } A = 2$.

② $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$

$\text{rank } A = \# \text{ non zero diag. entries.}$

Some more properties

Theorem Let A, B be $n \times n$ matrices.

- (1) $\text{rank } A = \text{rank } A^T$
- (2) $\det A \neq 0 \iff \text{rank } A = n$ (full rank)
- (3) $\text{rank}(AB) \leq \min(\text{rank } A, \text{rank } B)$
- (4) If A is invertible, i.e., $\det A \neq 0$, then
 $\text{rank}(AB) = \text{rank } B$,
 $\text{rank}(BA) = \text{rank } B$.
- (5) $\text{rank } A = \text{rank } B \iff \exists$ invertible P, Q .
 s.t. $A = PBQ$.
- (6) $\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n$.

Rank-one matrices

Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$. Denote $n \times n$ matrix

$$\vec{a} \otimes \vec{b} = [a_i b_j]_{n \times n} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \dots & \dots & \dots & \dots \\ a_n b_1 & a_n b_2 & \dots & a_n b_n \end{bmatrix}$$

Then ~~$a \otimes b$~~ $\text{rank}(\vec{a} \otimes \vec{b}) = 1$, if $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

Theorem If A is an $n \times n$ matrix, then
 A is a rank-1 matrix $\Leftrightarrow A = \vec{a} \otimes \vec{b}$
 for some nonzero vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$.

Proof Exercise. Q.E.D.

Rank-one matrices have some nice properties.

Examples (1) If $A = [a_{ij}]_{n \times n}$ is $n \times n$ matrix
 then $A = \sum_{i,j=1}^n a_{ij} e_i \otimes e_j$

where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component}$

(2) If A is $n \times n$, I is $n \times n$ identity matrix,
 $a, b \in \mathbb{R}^n$. then

$$\det(I + a \otimes b) = 1 + a \cdot b.$$

(3) If A, B are $n \times n$ matrices, $\det A = \det B \neq 0$ and $A - B$ is a rank-one matrix, then $\text{Cof} A - \text{Cof} B$ is also a rank-one matrix.

Proof

$$A - B = a \otimes b$$

$$A^{-1}(A - B)B^{-1} = A^{-1}(a \otimes b)B^{-1}$$

$$B^{-1} - A^{-1} = (A^{-1}a) \otimes (B^{-1}b)$$

$$\frac{(\text{Cof} B)^T}{\det B} - \frac{(\text{Cof} A)^T}{\det A} = (A^{-1}a) \otimes (B^{-1}b)$$

But $\det A = \det B \neq 0$. So,

$$(\text{Cof} B)^T - (\text{Cof} A)^T = (\det A) A^{-1}a \otimes B^{-1}b$$

$$\text{Cof} A - \text{Cof} B = (-B^{-1}b) \otimes ((\det A) A^{-1}a)$$

$$= (-B^{-1}b) \otimes (\text{Cof} A)^T a.$$

Q.E.D.

Hadamard jump condition

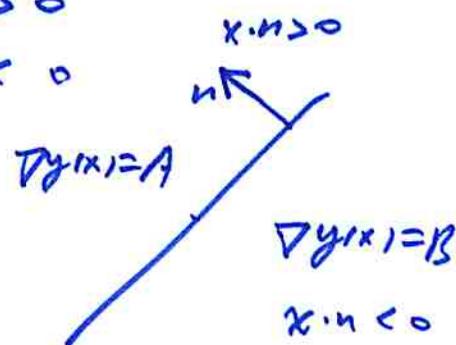
Let $n \in \mathbb{R}^3$ be a unit vector. There exists a continuous mapping (or deformation) $y: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\nabla y(x) = \begin{cases} A & \text{if } x \cdot n > 0 \\ B & \text{if } x \cdot n < 0 \end{cases}$$

for some matrices A, B .

$$\iff A - B = a \otimes n$$

for some $a \in \mathbb{R}^3$



Proof A good exercise!