

Math 210 Lecture Notes Bo Li, Fall 2013

Chapter 2. Matrix Techniques

Section 2.2 Eigenvalues and Eigenvectors

Definition Let A be an $n \times n$ matrix, $\vec{u} \in \mathbb{C}^n$, $\vec{u} \neq \vec{0}$, $\lambda \in \mathbb{C}$. Call λ an eigenvalue of A , and \vec{u} a corresponding eigenvector of A , if

$$A\vec{u} = \lambda\vec{u}$$

$$(\vec{u} \neq \vec{0}).$$

$$A\vec{u} - \lambda\vec{u} = \vec{0} \quad (A - \lambda I)\vec{u} = \vec{0}$$

Fact: Consider $B\vec{x} = \vec{0}$. If $\det B \neq 0$
 Then $B^{-1}B\vec{x} = B^{-1}\vec{0} = \vec{0}$. \uparrow
 $\Rightarrow \vec{x} = \vec{0}$. So, if $\vec{x} \neq \vec{0}$, then $\det B = 0$.

Thm λ is an eigenvalue of $A \Leftrightarrow \boxed{\det(A - \lambda I) = 0}$
 Called characteristic equation \uparrow

Two questions: (1) Diagonalization: When there exists a nonsingular ~~nonzero~~ matrix T such that
 (i.e., $\det T \neq 0$)

$$T^{-1}AT = \text{a diagonal matrix}$$

(2) How eigenvalues are related to entries of A (invariance, trace, determinant, ...).

Example $\begin{cases} x_1'(t) = x_1(t) + x_2(t) \\ x_2'(t) = 4x_1(t) + x_2(t). \end{cases}$ or: $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2. \end{cases}$

or $x' = Ax$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$

Find eigenvalues and eigenvectors.

$$\det(A - \lambda I) = 0 \quad \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = -1.$$

$\boxed{\lambda_1 = 3:} \quad A\vec{u}_1 = \lambda_1 \vec{u}_1. \quad (A - \lambda_1 I)\vec{u}_1 = \vec{0}.$

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \cancel{\text{---}} \text{---}$$

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad -2\xi_1 + \xi_2 = 0. \quad \xi_2 = +2\xi_1$$

$$\xi_1 = 1, \quad \xi_2 = 2. \quad \text{So, } \boxed{\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\boxed{\lambda_2 = -1} \quad A - \lambda_2 I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$$2\xi_1 + \xi_2 = 0. \quad \boxed{\vec{u}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

\vec{u}_1, \vec{u}_2 : linearly independent!

$$\text{Let } D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

$$T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

"Easy" to compute: $T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

check: $T^{-1}AT = D$. A is diagonalizable.

Let $x = Ty$, $y = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

$$x' = Ax \Rightarrow Ty' = ATy \Rightarrow y' = T'ATy = Dy$$

$\boxed{y' = Dy}$ i.e., $\begin{cases} y'_1 = 3y_1 \\ y'_2 = -y_2 \end{cases}$ Decoupled!

$$y_1 = c_1 e^{3t}, \quad y_2 = c_2 e^{-t}, \quad x = Ty \Rightarrow x_1 = \dots, \quad x_2 = \dots$$

In general, A is diagonalizable if $T^{-1}AT = D$ a diagonal matrix with $\det T \neq 0$.

$$T^{-1}AT = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ 0 & 0 & \ddots & \lambda_n \end{bmatrix} \Leftrightarrow AT = TD$$

Let $T = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$. Then $AT = TD$

$$\Leftrightarrow [A\vec{u}_1 \ A\vec{u}_2 \ \dots \ A\vec{u}_n] = [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \dots \ \lambda_n \vec{u}_n]$$

$$\text{Fill in: } TD = T[\lambda_1 e_1 \ \lambda_2 e_2 \ \dots \ \lambda_n e_n]$$

$$= [T\lambda_1 e_1 \ T\lambda_2 e_2 \ \dots \ T\lambda_n e_n]$$

$$= [\lambda_1 T e_1 \ \lambda_2 T e_2 \ \dots \ \lambda_n T e_n]$$

$$= [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \dots \ \lambda_n \vec{u}_n]$$

$$\Leftrightarrow \vec{u}_1 \ A\vec{u}_1 = \lambda_1 \vec{u}_1, \quad A\vec{u}_2 = \lambda_2 \vec{u}_2, \quad \dots, \quad A\vec{u}_n = \lambda_n \vec{u}_n.$$

A has n eigenvalues (possibly repeated, complex valued, etc.) and n eigenvectors $\vec{u}_1, \dots, \vec{u}_n$. So, $T = [\vec{u}_1 \ \dots \ \vec{u}_n]$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ 0 & 0 & \ddots & \lambda_n \end{bmatrix}$.

A is diagonalizable? Not really. We require that T be nonsingular: $\det T \neq 0$.

Theorem An $n \times n$ matrix A is diagonalizable
 $\Leftrightarrow A$ has n linearly independent eigenvectors.

Q.E.D.

$$T^{-1}AT = D$$

$$T = [\vec{u}_1 \dots \vec{u}_n]$$

$$D = \text{diag}(d_1, \dots, d_n)$$

d_1, \dots, d_n : eigenvalues

$\vec{u}_1, \dots, \vec{u}_n$: eigenvectors (linearly indep.)

Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable!

Proof. $\det[A - \lambda I] = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$.

So, $D = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If A is diagonalizable, then there exists invertible (i.e., non singular) matrix T s.t. $T^{-1}AT = D = I$.

$\Rightarrow A = TIT^{-1} = I$. Contradiction!

Definition Two $n \times n$ matrices A and B are similar, if there exists a nonsingular $n \times n$ matrix T such that

$$T^{-1}AT = B$$

We write $A \sim B$ to denote that A is similar to B .

So, A is diagonalizable means A is similar to a diagonal matrix.

Note that $A \sim A$
 $A \sim B \Rightarrow B \sim A$
 $A \sim B, B \sim C \Rightarrow A \sim C.$

Suppose $A \sim B$ with $T^{-1}AT = B$ ($\det T \neq 0$).

Hence Then $\lambda I - T^{-1}AT = \lambda I - B$

$$T^{-1}(\lambda I - A)T = \lambda I - B$$

$$\det(T^{-1}(\lambda I - A)T) = \det(\lambda I - B)$$

$$\det T^{-1} \cdot \det(\lambda I - A) \cdot \det T = \det(\lambda I - B)$$

$$(\det T)^{-1} \det(\lambda I - A) \det T = \det(\lambda I - B)$$

So,

$$\det(\lambda I - A) = \det(\lambda I - B)$$

$f_A(\lambda) = \det(\lambda I - A)$: the characteristic polynomial of A .

Theorem $A \sim B \Rightarrow f_A(\lambda) = f_B(\lambda)$. A.R.D.

In particular, $A \sim B \Rightarrow A, B$ have the same eigenvalues.

But, the reverse \Rightarrow is not true in general.
 See Example on previous page.

Definition Let A be an $n \times n$ matrix. The trace of A is the sum of all eigenvalues of A .

$$\text{tr } A = \lambda_1 + \dots + \lambda_n.$$

Thm

If $A \sim B$ with $T^{-1}AT = B$ ($\det T \neq 0$) then

$$(1) \quad \text{tr } A = \text{tr } B$$

$$(2) \quad \det A = \det B.$$

Q.E.D.

We thus say that trace and determinants are invariant w.r.t. similarity.

Now, look closer:

$\lambda_1, \dots, \lambda_n$: eigenvalues

$$f_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) + p_{n-2}(\lambda)$$

(p_{n-2} is a polynomial of $\deg \leq n-2$)

$$= \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots.$$

Hence $\lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots$$

So

$$\boxed{\begin{aligned} \text{trace } A &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ &= a_{11} + a_{22} + \cdots + a_{nn} \end{aligned}}$$

$$\det A = ? \quad f_A(0) = \det(-A) = (-1)^n \det A$$

$$f_A(0) = (0 - \lambda_1)(0 - \lambda_2) \cdots (0 - \lambda_n) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\boxed{\det A = \lambda_1 \lambda_2 \cdots \lambda_n}$$

A more general result:

Theorem. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let the characteristic polynomial of A be written as

$$\begin{aligned} f_A(\lambda) &= \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\ &= \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{n-1} \lambda + c_n \end{aligned}$$

Then for $1 \leq m \leq n$,

$$(-1)^m c_m = \sum_{i_1 < i_2 < \cdots < i_m} \Delta(i_1, i_2, \dots, i_m)$$

where $\Delta(i_1, i_2, \dots, i_m)$ is the $m \times m$ determinant formed from rows $i = i_1, i_2, \dots, i_m$ and columns $j = j_1, j_2, \dots, j_m$.

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\det(\lambda I - A) = \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3$

$$(-1)^1 c_1 = 1 + 5 + 9 = 15, \quad [c_1 = -15]$$

$$(-1)^2 c_2 = \Delta(1, 2) + \Delta(1, 3) + \Delta(2, 3)$$

$$= \left| \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \right| + \left| \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \right| + \left| \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \right| = -18. \quad [c_2 = -18]$$

$$(-1)^3 c_3 = \Delta(1, 2, 3) = \det A = 0. \quad [c_3 = 0]$$

Back to matrix diagonalization.

- (1) Diagonalization is useful.
- (2) A is diagonalizable $\Leftrightarrow A$ has n linearly independent eigenvectors
- (3) But not all matrices are diagonalizable.

Theorem Let A be an $n \times n$ matrix. If all n eigenvalues of A are distinct, then A is diagonalizable.

Proof. Suppose $A\vec{u}_k = \lambda_k \vec{u}_k$, $\vec{u}_k \neq \vec{0}$, $\lambda_1, \dots, \lambda_n$: distinct. We prove $\vec{u}_1, \dots, \vec{u}_n$ are linearly indeps. by induction. $\vec{u}_1 \neq \vec{0} \Rightarrow \{\vec{u}_1\}$ is linearly indeps.

Assume $\{\vec{u}_1, \dots, \vec{u}_k\}$ is L. I. (L. I.)

Show that $\{\vec{u}_1, \dots, \vec{u}_k, \vec{u}_{k+1}\}$ is L. I.

$$\text{Let } c_1 \vec{u}_1 + \dots + c_k \vec{u}_k + c_{k+1} \vec{u}_{k+1} = \vec{0} \quad (*)$$

$$\text{So, } A(c_1 \vec{u}_1 + \dots + c_k \vec{u}_k + c_{k+1} \vec{u}_{k+1}) = \vec{0}$$

$$c_1 A \vec{u}_1 + \dots + c_k A \vec{u}_k + c_{k+1} A \vec{u}_{k+1} = \vec{0}$$

$$c_1 \lambda_1 \vec{u}_1 + \dots + c_k \lambda_k \vec{u}_k + c_{k+1} \lambda_{k+1} \vec{u}_{k+1} = \vec{0}$$

$$\lambda_{k+1} (*) \Rightarrow c_1 \lambda_{k+1} \vec{u}_1 + \dots + c_k \lambda_{k+1} \vec{u}_k + c_{k+1} \lambda_{k+1} \vec{u}_{k+1} = \vec{0}$$

$$\text{Hence. } c_1 (\lambda_{k+1} - \lambda_1) \vec{u}_1 + \dots + c_k (\lambda_{k+1} - \lambda_k) \vec{u}_k = \vec{0}$$

But, $\vec{u}_1, \dots, \vec{u}_k$ are L. I. Hence $c_1 (\lambda_{k+1} - \lambda_1) = 0$

$$c_2 (\lambda_{k+1} - \lambda_2) = 0, \dots, c_k (\lambda_{k+1} - \lambda_k) = 0$$

But λ_j are distinct. So, all $c_1 = c_2 = \dots = c_k = 0$

$$\text{So, } c_{k+1} \vec{u}_{k+1} = \vec{0} \Rightarrow c_{k+1} = 0. \quad \underline{\text{Q.E.D.}}$$