

Section 2.4 Matrix Exponentials

Motivation. If  $x = x(t)$  satisfy  $\begin{cases} x'(t) = a x(t) \\ x(0) = x_0 \end{cases}$   
then  $x(t) = x_0 e^{at}$

Generalization:  $\begin{cases} \bar{x}' = A\bar{x} \\ \bar{x}(0) = \bar{x}_0 \end{cases}$   $A: n \times n, \bar{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

e.g.  
 $x_1' = -5x_1 + 2x_2$   
 $x_2' = -6x_1 + 2x_2$

has sol'n  $\bar{x}(t) = \underbrace{\bar{x}_0 e^{At}}_{\text{not good in dimension}} = e^{At} \bar{x}_0$

Definition  $A: n \times n, e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{A^k}{k!} (A^0 = I.)$

Convergent? Yes.

Let  $A = [a_{ij}]_{n \times n}$ . Let  $a = \max_{1 \leq i, j \leq n} |a_{ij}|$ .

Then  $|(A^2)_{ij}| = |\sum_{k=1}^n a_{ik} a_{kj}| \leq n a^2$

Suppose  $|(A^k)_{ij}| \leq n^{k-1} a^k$ . Then

$$\begin{aligned} |(A^{k+1})_{ij}| &= |(A^k \cdot A)|_{ij} = \left| \sum_{k=1}^n (A^k)_{ijk} a_{kj} \right| \\ &\leq n \cdot n^{k-1} a^k \cdot a = n^k a^{k+1}. \end{aligned}$$

But,  $\sum_{k=0}^{\infty} \frac{n^k a^{k+1}}{k!}$  converges.

So, for each  $(i, j)$ ,  $\sum_{k=0}^{\infty} \frac{(A^k)_{ij}}{k!}$  converges ABSOLUTELY.

How to compute  $e^A$ ?

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Example 1  $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$   $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^k}{k!} & & \\ & \ddots & \\ & & \frac{\lambda_n^k}{k!} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}$$

$$\text{So, } e^{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}} = \begin{bmatrix} e & 0 \\ 0 & e^{-2} \end{bmatrix}$$

Example 2 Suppose  $A$  is diagonalizable:

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$\lambda_1, \dots, \lambda_n$ : eigenvalues of  $A$

Columns of  $S$ : (linearly independent) eigenvectors of  $A$ .

Then

$$A = S\Lambda S^{-1}$$

$$A^2 = S\Lambda S^{-1} \cdot S\Lambda S^{-1} = S\Lambda(S^{-1}S)\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

$$A^3 = S\Lambda^3 S^{-1}$$

$$A^k = S\Lambda^k S^{-1} = S \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} S^{-1}$$

$$A^0 + A + \frac{1}{2!}A^2 + \dots + \frac{1}{n!}A^n$$

$$= I + S\Lambda S^{-1} + \frac{1}{2!}S\Lambda^2 S^{-1} + \dots + \frac{1}{n!}S\Lambda^n S^{-1}$$

$$= S I S^{-1} + S\Lambda S^{-1} + S \left(\frac{1}{2!}\Lambda^2\right) S^{-1} + \dots + S \left(\frac{1}{n!}\Lambda^n\right) S^{-1}$$

$$= S \left( I + \Lambda + \frac{1}{2!}\Lambda^2 + \dots + \frac{1}{n!}\Lambda^n \right) S^{-1}$$

$$\rightarrow S \left( \sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k \right) S^{-1} = S e^{\Lambda} S^{-1} = S \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} S^{-1}$$

We have in fact proved:

Theorem ① If  $S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$  ( $\det S \neq 0$ ). Then

$$e^A = S e^\Lambda S^{-1} = S \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} S^{-1}$$

② If  $S^{-1}AS = B$  then  $e^A = S e^B S^{-1}$ .

Example  $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ . Symmetric! So, diagonalizable.

$$S^{-1}AS = \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$e^A = S e^\Lambda S^{-1} = \dots = \frac{1}{2} \begin{bmatrix} e^{-1} + e^{-3} & e^{-1} - e^{-3} \\ e^{-1} - e^{-3} & e^{-1} + e^{-3} \end{bmatrix}$$

Method 1: Diagonalization! (Not always work.  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is not diagonalizable.)

Method 2: Use The Cayley-Hamilton Theorem.

Thm (Cayley-Hamilton) Let  ~~$f(x)$~~   $f(\lambda) = \det(\lambda I - A)$ .

Then  $f(A) = 0$ .

Bad notation!

More precisely:  $f(\lambda) = \det(\lambda I - A) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$ .

Then  $A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = 0$

(The  $n \times n$  zero matrix)

Note that this implies  $A^n = -a_{n-1} A^{n-1} - \dots - a_0 I$

$$A^{n+1} + a_{n-1} A^n + \dots + a_1 A^2 + a_0 A = 0$$

$$\Rightarrow A^{n+1} = \square A^{n-1} + \square A^{n-2} + \dots + \square I$$

So, all  $A^k$  ( $k \geq n$ ) are linear combinations of  $A^{n-1}, A^{n-2}, \dots, A, I$ .

Thus,  $\sum_0^N A^k$  is also such a linear combination.

Finally,  $e^A$  is, too. So,

$$e^A = d_{n-1} A^{n-1} + d_{n-2} A^{n-2} + \dots + d_0 I.$$

Need to determine  $d_0, \dots, d_{n-1}$  ( $n$  numbers)

Now, let  $\lambda_1, \dots, \lambda_n$  be eigenvalues of  $A$ , then we have similarly

$$e^{\lambda_j} = d_{n-1} \lambda_j^{n-1} + \dots + d_0, \quad j=1, 2, \dots, n.$$

If  $\lambda_j$  are distinct, then we can solve for  $d_{n-1}, \dots, d_0$ .

If not all distinct, we use some other equations.

e.g., if  $\lambda_1$  is a repeated root of  $f(\lambda)$ , then

$$f'(\lambda) = 0. \quad \text{So, } e^{\lambda_1} = \frac{d}{d\lambda} (d_{n-1} \lambda^{n-1} + \dots + d_1 \lambda + d_0) \Big|_{\lambda=\lambda_1}.$$

Complex eigenvalues can also be treated.

Example  $A = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix}, \quad \lambda_1 = -1, \lambda_2 = -2.$

$$e^A = d_1 A + d_0 I.$$

$$e^{\lambda_j} = d_1 \lambda_j + d_0 \quad (j=1, 2) \quad \left\{ \begin{array}{l} e^{-1} = d_1(-1) + d_0 \\ e^{-2} = d_1(-2) + d_0 \end{array} \right.$$

$$d_1 = e^{-1} - e^{-2}, \quad d_0 = 2e^{-1} - e^{-2}$$

$$e^A = (e^{-1} - e^{-2}) \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} + (2e^{-1} - e^{-2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3e^{-1} + 4e^{-2} & 2e^{-1} - 2e^{-2} \\ 6e^{-2} - 6e^{-1} & 4e^{-1} - 3e^{-2} \end{bmatrix}.$$

Note:  $f_A(\lambda) = \det \begin{bmatrix} \lambda + 5 & -2 \\ 6 & \lambda - 2 \end{bmatrix} = \lambda^2 + 3\lambda + 2.$

Cayley-Hamilton:  $A^2 + 3A + 2I = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} + 3 \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -6 \\ 18 & 8 \end{bmatrix} + \begin{bmatrix} -15 & 6 \\ -18 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Method 3. Use Jordan decomposition (or Jordan form) 5  
 (canonical form of matrix).

Call  $J = \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}$  a Jordan block.

e.g.  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ .

Example.  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ,  $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Now, you can guess:  $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & ? \\ 0 & e \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$$

$$? = \sum_{k=0}^{\infty} \frac{1}{k!} k = \sum_{k=1}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = \sum_{j=0}^{\infty} \frac{1}{j!} = e$$

$$e^A = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$$

Example  $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$   $3 \times 3$ .  $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

$$e^J = \begin{bmatrix} e^\lambda & e^\lambda & \frac{1}{2}e^\lambda \\ 0 & e^\lambda & e^\lambda \\ 0 & 0 & e^\lambda \end{bmatrix}$$

Theorem If  $A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$  then  $e^A = \begin{bmatrix} e^{A_1} & & 0 \\ & \ddots & \\ 0 & & e^{A_m} \end{bmatrix}$

$A_1, \dots, A_m$ : blocks. (square matrices).

Reason:  $A^k = \begin{bmatrix} A_1^k & & \\ & \ddots & \\ & & A_m^k \end{bmatrix}$ .

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Theorem (Jordan Canonical Form) Let  $A$  be an  $n \times n$  matrix with  $s$  distinct eigenvalues  $\lambda_1, \dots, \lambda_s$  such that

$$f_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_s)^{m_s}$$

( $m_i \geq 1$ : integer). Then  $A$  is similar to

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_s \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

With this,  $M^{-1} A M = J$ ,  $A = M J M^{-1}$   
 $e^A = M e^J M^{-1}$ ,  $e^J = \begin{bmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_s} \end{bmatrix}$ .

Properties of matrix exponentials

(1)  $e^0 = I$ .

(2)  $AB = BA \Rightarrow e^A \cdot e^B = e^{A+B} = e^B \cdot e^A$ .

Pf.  $e^A e^B = \left( \sum_0^{\infty} \frac{1}{k!} A^k \right) \left( \sum_0^{\infty} \frac{1}{k!} B^k \right)$

$$= \sum_{k=0}^{\infty} C_k$$

$$C_k = \frac{1}{k!} A^k + \frac{1}{(k-1)!} A^{k-1} B + \frac{1}{(k-2)!} A^{k-2} \frac{1}{2!} B^2$$

$$+ \dots + \frac{1}{k!} A B^k + \frac{1}{k!} B^k$$

$$= \frac{1}{k!} \left( A^k + \binom{k}{k-1} A^{k-1} B + \binom{k}{k-2} A^{k-2} B^2 + \dots + B^k \right)$$

$$= \frac{1}{k!} (A+B)^k \quad \text{Q.E.D.}$$

(3)  $e^A$  is always invertible. In fact,  $(e^A)^{-1} = e^{-A}$ .

Pf  $e^{-A} \cdot e^A = e^{-A+A} = e^0 = I$ . Since  $(A)A = A(-A)$ .  
Use (1), (2). Q.E.D.

(4)  $e^{A^T} = (e^A)^T$

Pf  $e^{A^T} = \lim_{N \rightarrow \infty} \sum_0^N \frac{1}{k!} (A^T)^k = \lim_{N \rightarrow \infty} \left( \sum_0^N \frac{1}{k!} A^k \right)^T$

$= \left( \lim_{N \rightarrow \infty} \sum_0^N \frac{1}{k!} A^k \right)^T = (e^A)^T$  Q.E.D.

(5)  $A^T = -A$  (i.e.,  $A$  is skew-symmetric), then  $e^A$  is orthogonal. (exercise)

(6)  $e^{S^{-1}AS} = S^{-1}e^AS$  (proved before)

(7)  $\det e^A = e^{\text{tr}(A)} > 0$  (This  $\Rightarrow$  (3))

Pf Jordan:  $M^{-1}AM = J$ .  $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$   
 $J_i$ : Jordan blocks.  
 $J_i$ :  $m_i \times m_i$ ,  $\lambda_i$ .

$A = M J M^{-1}$

$e^A = M e^J M^{-1}$

$\det e^A = \det M \cdot \det e^J \cdot \det M^{-1} = \det e^J$

$= \det \begin{bmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_s} \end{bmatrix} = \det e^{J_1} \dots \det e^{J_s}$

$J_i = \begin{bmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}_{m_i \times m_i}$   $\text{tr } J_i = \underbrace{\lambda_i + \dots + \lambda_i}_{m_i} = m_i \lambda_i$

$\text{tr } A = \text{tr } J = \sum_{i=1}^s \text{tr } J_i = \sum_{i=1}^s m_i \lambda_i = \sum_{i=1}^s \text{tr } J_i$ .

Only need to show

$\det e^{J_i} = e^{\text{tr } J_i}$

$e^{J_i} = \begin{bmatrix} e^{\lambda_i} & & \\ & \ddots & \\ 0 & & e^{\lambda_i} \end{bmatrix}_{m_i \times m_i}$

$= e^{m_i \lambda_i} = e^{\text{tr } J_i}$

So,  $\det e^{J_i} = \underbrace{e^{\lambda_i} \dots e^{\lambda_i}}_{m_i}$

Q.E.D.

Now, some properties of the matrix-valued function  $e^{At}$  ( $= e^{tA}$ ).

①  $e^{At} \cdot e^{At'} = e^{A(t+t')}$

②  $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$ . by: unif. convergence

Pf.  $\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_0^{\infty} \frac{1}{k!} A^k t^k = \sum_1^{\infty} \frac{1}{(k-1)!} A^k k t^{k-1}$   
 $= A \sum_1^{\infty} \frac{1}{(k-1)!} A^{k-1} t^{k-1} = Ae^{At}$ . Q.E.D.

Thm.  $\begin{cases} \dot{\vec{x}} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$  has the sol'n  $\vec{x}(t) = e^{At} \vec{x}_0$ .

Pf. Verify:  $\vec{x}(0) = e^{A \cdot 0} \vec{x}_0 = I \vec{x}_0 = \vec{x}_0$   
 $\dot{\vec{x}} = \frac{d}{dt} (e^{At} \vec{x}_0) = (e^{At})' \vec{x}_0 = Ae^{At} \vec{x}_0 = A\vec{x}$ . Q.E.D.

Example.  $e^A \cdot e^B \neq e^{A+B}$  in general

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$   $B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ .  $A = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

$e^A e^B = \frac{1}{2} \begin{bmatrix} 1+e^2 & 1-e^2 \\ 1-e^2 & 1+e^2 \end{bmatrix} \begin{bmatrix} e^{-1} & 1-e^{-1} \\ 0 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} e^{-1}+e & 2-e^{-1}-e \\ e^{-1}-e & 2-e^{-1}+e \end{bmatrix}$

$e^{A+B} = \begin{bmatrix} 1 & 0 \\ 1-e & e \end{bmatrix}$ .

Since  $AB \neq BA$ .

Example In general.

$$\frac{d}{dt} e^{B(t)} \neq \frac{d}{dt} B(t) e^{B(t)}$$

$$\frac{d}{dt} e^{B(t)} \neq e^{B(t)} \frac{d}{dt} B(t).$$

Try  $B(t) = \begin{bmatrix} t & 0 \\ t^2 & -t \end{bmatrix}$ .

More examples

Example Solve  $\dot{x} = Ax$ .  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x(t)$ .  $A = \begin{bmatrix} 6 & 3 & -2 \\ -4 & -1 & 2 \\ 13 & 9 & -3 \end{bmatrix}$ .  
 $x(0) = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

Solution.

$$P^{-1}AP = \Lambda = \begin{bmatrix} 1 & & \\ & 2 & \\ & & -1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -1 & \sqrt{2} \\ -1 & 2 & -\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = P \Lambda P^{-1} \quad A t = P \Lambda t P^{-1}$$

$$e^{At} = P e^{\Lambda t} P^{-1} = \begin{bmatrix} 1 & -1 & \sqrt{2} \\ -1 & 2 & -\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 0 \\ -6 & -4 & 2 \end{bmatrix}$$

$$x = e^{At} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \dots = \begin{bmatrix} -11e^t + e^{2t} + 8e^{-t} \\ 11e^t - 2e^{2t} - 8e^{-t} \\ -11e^t - e^{2t} + 16e^{-t} \end{bmatrix}$$

Example  $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$   $a, b \in \mathbb{R}$ .

$$e^B = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

In fact,  $B = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = aI + bJ$

$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  (not a Jordan block).

$$e^B = e^{aI + bJ} = e^{aI} \cdot e^{bJ} \quad (\text{since } (aI)(bJ) = (bJ)(aI))$$

Why  $e^{bJ} = \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$ ?

Let  $A = J$ .  
 Solve  $\dot{x} = Ax$   
 $x(0) = x_0$ .  
 Soln  $x = e^{At} x_0$ .

$$\begin{cases} x' = Jx \\ x(0) = x_0 \end{cases} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(\lambda I - J) = \det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\lambda_1 = i: \quad iI - J = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

$$i u_1 + u_2 = 0. \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -i u_1 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\lambda_2 = -i. \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & i \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

But, need real.

$$\begin{aligned} e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} &= (\cos t + i \sin t) \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \\ &= \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &\quad + i \left( \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \end{aligned}$$

Two linearly indep. solns:  $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

The general soln  $x = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

$$x_0 = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix} \Rightarrow x = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$x_0 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = -1 \end{matrix} \Rightarrow x = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

$$x = e^{Jt} x_0 \Rightarrow e^{Jt} = e^{Jt} I = e^{Jt} (e_1, e_2) = [e^{Jt} e_1, e^{Jt} e_2]$$

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \quad \text{Conclusion} \\ e^{\begin{bmatrix} a & -b \\ b & a \end{bmatrix} t} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$