

Chapter 3 Hilbert Spaces

Section 3.1 Definition, Properties, and Examples

Section 3.2 Best Approximations, Orthogonality

Section 3.3 Orthogonal Polynomials

Section 3.4 Linear Functionals and Linear Operators

Section 3.1 Definition, Properties, and Examples

We reviewed inner product and orthogonality of \mathbb{R}^n in section 2.3. We now give a general definition of an inner product.

We denote by \mathbb{F} the field \mathbb{R} (real) or \mathbb{C} (complex).

$$\mathbb{R} = \{\text{all real numbers}\}$$

$$\mathbb{C} = \{\text{all complex numbers}\}$$

$$\text{So, } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}.$$

Recall: X is a vector space

over \mathbb{F} means $X (\neq \emptyset)$

has two operations:

addition $x+y$ ($x, y \in X$)

and scalar multiplication

αx ($\alpha \in \mathbb{F}, x \in X$).

They satisfy 8 properties

$$x+y = y+x$$

$$x+(y+z) = (x+y)+z$$

$$\exists 0: 0+x = x$$

$$\exists -x: (-x)+x = 0$$

$$\alpha(\beta x) = (\alpha\beta)x$$

$$(\alpha+\beta)x = \alpha x + \beta x$$

$$\alpha(x+y) = \alpha x + \alpha y$$

$$1x = x$$

$$3-4i \in \mathbb{C}$$

$$3+4i, 3-4i$$

conjugate numbers.

3 is the real part
and 4 is the imaginary part of $3+4i$.

$$|3+4i| = \sqrt{3^2+4^2} = 5.$$

Examples of vector spaces:

$$\mathbb{R}^n, \mathbb{C}^n, C([a,b])$$

P_n (polynomials of $\deg. \leq n$)

$M^{m \times n}$ (matrices)

P all polynomials

$L^2(a,b), \dots$

Definition (Inner product) Let X be a vector space over \mathbb{F} . Suppose for any $u, v \in X$ there is a number $u \cdot v$ or $\langle u, v \rangle$ in \mathbb{F} , such that

(1) For any $u \in X$, $\langle u, u \rangle \geq 0$. (number 0)

$$\langle u, u \rangle = 0 \iff u = 0 \text{ (vector } 0 \text{ in } X\text{)}$$

(2) $\langle u, v \rangle = \langle v, u \rangle$ if $\mathbb{F} = \mathbb{R}$ $\forall u, v \in X$

$$\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \text{if } \mathbb{F} = \mathbb{C} \quad \text{This one is more general!}$$

(3) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$$\forall u, v, w \in X$$

$$(4) \quad \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \quad \forall u, v \in X \quad \forall \alpha \in \mathbb{F}.$$

Then we call $\langle u, v \rangle$ the inner product of $u, v \in X$. In this case (i.e., we have an inner product on X), we call X an inner product space.

Examples \mathbb{R}^n is a real vector space, i.e., a vector space over \mathbb{R} . If $u = [u_1 \atop \vdots \atop u_n]$, $v = [v_1 \atop \vdots \atop v_n] \in \mathbb{R}^n$, then

$$\langle u, v \rangle = \sum_{j=1}^n u_j v_j \quad (\text{a real number})$$

defines an inner product of u, v in \mathbb{R}^n .

\mathbb{C}^n is a complex vector space, i.e., a vector space over \mathbb{C} . If $u = [u_1 \atop \vdots \atop u_n]$, $v = [v_1 \atop \vdots \atop v_n] \in \mathbb{C}^n$ then

$$\langle u, v \rangle = \sum_{j=1}^n u_j \overline{v_j} \quad (\text{a complex number})$$

defines an inner product of u, v in \mathbb{C}^n . Note here we have $\overline{v_j}$ instead of v_j . For example,

$$\begin{aligned} n=3: \quad \begin{bmatrix} i \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1+2i \\ 3i \\ 2-i \end{bmatrix} &= i \cdot \overline{(-1+2i)} + 0 \cdot \overline{3i} + (-1) \overline{(2-i)} \\ &= i(-1-2i) - (2+i) \\ &= -i + 2 - 2 - i = -2i \quad \boxed{-2i} \end{aligned}$$

$$\boxed{i \cdot i = -1}$$

Some immediate properties of inner product, following the definition. This is the 0 vector in X

$$(1) \quad \langle 0, u \rangle = 0, \quad \langle u, 0 \rangle = 0 \quad \forall u \in X.$$

This is the number 0.

Proof $\langle 0, u \rangle = \langle d0, u \rangle = d\langle 0, u \rangle$. Set $d=0 \in F$.

$$\langle 0, u \rangle = 0 \langle 0, u \rangle = 0.$$

$$\langle u, 0 \rangle = \overline{\langle 0, u \rangle} = \bar{0} = 0. \quad \underline{\text{Q.E.D.}}$$

$$(2) \quad \langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle. \quad \forall u, v \in X, \alpha \in C (\text{or } R).$$

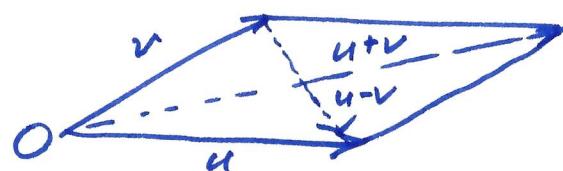
Proof $\langle u, \alpha v \rangle = \overline{\langle \alpha v, u \rangle} = \overline{\alpha \langle v, u \rangle} = \bar{\alpha} \overline{\langle v, u \rangle}$
 $= \bar{\alpha} \langle u, v \rangle. \quad \forall u, v \in X, \forall \alpha \in F \quad \underline{\text{Q.E.D.}}$
 $(F=C \text{ or } R)$

Notation $\|u\| = \sqrt{\langle u, u \rangle} \quad \forall u \in X$ (an inner product space)

The law of parallelogram In an inner product space X .

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

$$\begin{aligned} \text{Pf} \quad \|u+v\|^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u+v \rangle + \langle v, u+v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle \\ &\quad + \langle v, u \rangle + \langle v, v \rangle \end{aligned}$$



$$\begin{aligned} z &= a+bi \\ \bar{z} &= a-bi \\ (a, b : \text{real}) \\ z + \bar{z} &= 2a = 2Rez \end{aligned}$$

$$\text{Similarly, } \|u-v\|^2 = \|u\|^2 - 2\operatorname{Re}\langle u, v \rangle + \|v\|^2.$$

$$\text{So, } \|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2. \quad \underline{\text{Q.E.D.}}$$

Let X be an inner-product space. If $u, v \in X$ and $\langle u, v \rangle = 0$ then we say u and v are orthogonal. Sometimes denoted as $u \perp v$.

The Pythagorean Thm (勾股定理)

If $u \perp v$ then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Pf Same as previous calculation:

$$\|u+v\|^2 = \|u\|^2 + 2 \underbrace{\operatorname{Re} \langle u, v \rangle}_0 + \|v\|^2. \quad \underline{\text{Q.E.D.}}$$

The Cauchy-Schwarz inequality

$$|\langle u, v \rangle| \leq \|u\| \|v\| \quad \forall u, v \in X \text{ (an inner product space)}$$

The equality holds $\Leftrightarrow u, v$ are linearly dependent.

Pf. Assume, say, $u \neq 0$. Then for any complex (or real) number α , $\alpha \bar{\alpha} = |\alpha|^2$

$$0 \leq \|\alpha u - v\|^2 = |\alpha|^2 \|u\|^2 + \|v\|^2 - 2\alpha \langle u, v \rangle - \bar{\alpha} \langle u, v \rangle$$

Let $\alpha = \frac{\langle u, v \rangle}{\|u\|^2}$. Then

$$0 \leq \|\alpha u - v\|^2 = \frac{|\langle u, v \rangle|^2}{\|u\|^4} \|u\|^2 + \|v\|^2$$

$$- \frac{|\langle u, v \rangle|^2}{\|u\|^2} - \frac{|\langle u, v \rangle|^2}{\|u\|^2}$$

$$= \|u\|^2 - \frac{|\langle u, v \rangle|^2}{\|u\|^2}$$

$$\text{Hence, } \|u\|^2 \|v\|^2 \geq |\langle u, v \rangle|^2.$$

$$"=" \Leftrightarrow \alpha u = v. (\Leftrightarrow \|\alpha u - v\|^2 = 0). \quad \underline{\text{Q.E.D.}}$$

In an inner-product space X , for any vector $u \in X$, $\|u\| = \sqrt{\langle u, u \rangle}$ is the "length" of u . This "length" $\|u\|$ satisfies the following properties:

$$(1) \|u\| \geq 0, \|u\| = 0 \Leftrightarrow u = 0.$$

$$(2) \|\alpha u\| = |\alpha| \|u\| \quad \forall \alpha \in \mathbb{F} \quad \forall u \in X.$$

(3) (triangle inequality)

$$\|u+v\| \leq \|u\| + \|v\| \quad \forall u, v \in X.$$

Let's look at (3):

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle \leq \|u\|^2 + \|v\|^2 + 2\|u\|\cdot\|v\|.$$

True by the Cauchy-Schwarz inequality.
Note: (3) implies $\|\|u\| - \|v\|\| \leq \|u-v\|$.

We call $\|u\|$ the norm of u as $\|\cdot\|$ satisfies

(1)–(3). When a vector space X has a norm then it is called a normed vector space. An inner-product space is a normed vector space with the norm being induced by the inner-product

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

Definition Let X be an inner-product space.

(1) Let $u_n \in X$ ($n=1, 2, \dots$) and $u \in X$. If

$$\lim_{n \rightarrow \infty} \|u_n - u\| = 0$$

then we say $\{u_n\}$ converges to u in X and denote it $u_n \rightarrow u$ as $n \rightarrow \infty$ or simply $u_n \rightarrow u$ in X .

(2) Let $u_n \in X$ ($n=1, 2, \dots$). If $\lim_{n, m \rightarrow \infty} \|u_n - u_m\| = 0$ then we call $\{u_n\}$ a Cauchy sequence in X .
[$\forall \varepsilon > 0 \exists N. n, m \geq N \Rightarrow \|u_n - u_m\| < \varepsilon$].

Definition An inner-product space is complete if any Cauchy sequence converges in the space.

A complete inner-product space is called a Hilbert space.

Example Let $\mathbb{Q} = \{ \text{all rational numbers} \}$. A rational number is a number of the form $\frac{p}{q}$ where p, q are integers. \mathbb{Q} is a vector space over \mathbb{Q} (not over \mathbb{R} or \mathbb{C}). \mathbb{Q}

is an inner product space with $\langle a, b \rangle = ab$ the usual product of two numbers. Then, \mathbb{Q} is not complete,

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

$\lim_{n \rightarrow \infty} u_n = e \notin \mathbb{Q}$. $\{u_n\}$ is a Cauchy sequence.

That is a strange example.

We will have more ~~ex~~ examples. But first let us get familiar with ^{the} convergence.

Properties Let X be an inner-product space.

Suppose $u_n \rightarrow u$ and $v_n \rightarrow v$ in X . Suppose $\alpha_n \rightarrow \alpha$ in \mathbb{F} . Then

$$(1) \quad \lim_{n \rightarrow \infty} (u_n + v_n) = u + v \text{ in } X$$

$$(2) \quad \lim_{n \rightarrow \infty} \alpha_n u_n = \alpha u \text{ in } X$$

$$(3) \quad \lim_{n \rightarrow \infty} \|u_n\| = \|u\|$$

$$(4) \quad \lim_{n \rightarrow \infty} \langle u_n, v_n \rangle = \langle u, v \rangle$$

Notes: (1) and (2) mean that addition and scalar multiplication are continuous.
 (3) means $\|\cdot\|$ (the norm) is continuous.
 (4) means the inner product is a continuous function.

Pf (3) $|\|u_n\| - \|u\|| \leq \|u_n - u\| \rightarrow 0.$

(2) is similar to (4)

So, pf of (4)

$$\begin{aligned} & |\langle u_n, v_n \rangle - \langle u, v \rangle| \\ &= |\langle u_n - u, v_n \rangle + \langle u, v_n \rangle - \langle u, v \rangle| \\ &= |\langle u_n - u, v_n - v \rangle + \langle u_n - u, v \rangle + \langle u, v_n - v \rangle| \\ &\leq |\langle u_n - u, v_n - v \rangle| + |\langle u_n - u, v \rangle| + |\langle u, v_n - v \rangle| \\ &\leq \|u_n - u\| \|v_n - v\| + \|u_n - u\| \|v\| + \|u\| \|v_n - v\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Q.E.D.

Property If $u_n \rightarrow u$ in X then $\{u_n\}$ is a Cauchy sequence in X .

Pf Similar to sequence of numbers. Q.E.D.

Examples of Hilbert spaces

1. $\mathbb{R}^n = \{ u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} : u_1, \dots, u_n \in \mathbb{R} \}$. Inner product $u \cdot v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{j=1}^n u_j \cdot v_j$.
 $\|u\| = \sqrt{\sum u_j^2}$.

\mathbb{R}^n is a Hilbert space.

$$u^{(k)} = \begin{bmatrix} u_1^{(k)} \\ \vdots \\ u_n^{(k)} \end{bmatrix} \rightarrow u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \iff u_1^{(k)} \xrightarrow{(k)} u_1, \dots, u_n^{(k)} \xrightarrow{(k)} u_n$$

Convergence of vectors in $\mathbb{R}^n \iff$ convergence componentwise.

\mathbb{R} is complete: any Cauchy sequence of real numbers converges to a real number.

\mathbb{R}^n is complete, since each component is complete.

2. Similarly, \mathbb{C}^n is a Hilbert space.

Note again, if $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{C}^n$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{C}^n$ then the inner product is

$$\langle u, v \rangle = \sum_{j=1}^n u_j \overline{v_j}.$$

3. $L^2(a,b) = \{ \text{all square-integrable, complex-valued functions on } [a,b] \}$

$f \in L^2(a,b)$ means $\int_a^b |f(x)|^2 dx < \infty$.

Inner product: $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$

[If $g(x) = \cos x + i \sin x$ then $\overline{g(x)} = \cos x - i \sin x$]

Theorem $L^2(a,b)$ is a Hilbert space.

If $f_n \in L^2(a,b)$ ($n=1, 2, \dots$) and $f \in L^2(a,b)$, then $f_n \rightarrow f$ in $L^2(a,b)$ means precisely

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)|^2 dx = 0.$$

Note: In $L^2(a,b)$ any "vector" $f = f(x)$ such that $\int_a^b |f(x)|^2 dx = 0$ is considered to be the zero vector. Said differently, in $L^2(a,b)$, $f = g$ (two vectors f and g are the same) $\iff \int_a^b |f(x) - g(x)|^2 dx = 0$.
 $\quad (\iff \underbrace{f = g \text{ a.e. } x \in (a,b)}_{\text{Using the language of Lebesgue measure}})$

4. Let $P = \{ \text{all real polynomials} \}$.

Let $L^2(a,b)$ be the real Hilbert space with inner-product space.

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx.$$

Then, P is a subspace of $L^2(a,b)$. P itself is an inner product space with the same inner product as that in $L^2(a,b)$.

P is NOT complete. Why?

5. Let $\ell^2 = \{(a_1, a_2, \dots) : \text{all } a_j \in \mathbb{R} \text{ and } \sum_{j=1}^{\infty} a_j^2 < \infty\}$.

This is a real vector space.

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1+b_1, a_2+b_2, \dots)$$

$$c(a_1, a_2, \dots) = (ca_1, ca_2, \dots)$$

This is an inner-product space

$$(a_1, a_2, \dots) \cdot (b_1, b_2, \dots) = \sum_{j=1}^{\infty} a_j b_j.$$

$$\begin{aligned} \text{Note: } \sum_{j=1}^{\infty} a_j^2 < \infty, \sum_{j=1}^{\infty} b_j^2 < \infty &\Rightarrow \sum_{j=1}^{\infty} |a_j b_j| \\ &\leq \sum_{j=1}^{\infty} \frac{1}{2}(a_j^2 + b_j^2) < \infty. \end{aligned}$$

ℓ^2 is a Hilbert space.

Define the "coordinate vectors"

$$e_1 = (1, 0, 0, \dots)$$

$$e_2 = (0, 1, 0, \dots)$$

$$e_3 = (0, 0, 1, \dots)$$

.....

$$\boxed{\langle e_i, e_j \rangle = \delta_{ij}}$$

$$\begin{aligned} i \neq j: \|e_i - e_j\|^2 &= \langle e_i - e_j, e_i - e_j \rangle \\ &= \underbrace{\langle e_i, e_i \rangle}_{=1} - 2\langle e_i, e_j \rangle + \langle e_j, e_j \rangle = 1 + 1 = 2 \end{aligned}$$

$$\text{So, } \|e_i - e_j\| = \sqrt{2}, \quad i \neq j.$$

$\dim \ell^2 = \infty$ since all e_1, e_2, e_3, \dots are linearly independent (i.e., any finitely many of these vectors are linearly independent).

Convergence of vectors in $\ell^2 \Leftrightarrow$ convergence componentwise.