Math 210 Lecture Notes Bo Li, Fall 2013

## Section 3.3 Orthonormal Systems and Orthogonal ~~and Orthogonal Polynomials~~ Polynomials

**Definition** Let $X$ be a Hilbert space. Suppose $u_1, u_2, \cdots, u_n, \cdots$ are vectors of $X$ such that

$$\langle u_j, u_k \rangle = \delta_{jk} \quad (j, k = 1, 2, \cdots)$$

Then we call $\{u_j\}_{j=1}^{\infty}$ an orthonormal sequence of vectors or orthonormal system.

**Examples** ① $X = L^2(-\pi, \pi)$.

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \cdots \right\}$$

is an orthonormal system.

Since

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0.$$

If $\{u_j\}_{j=1}^{\infty}$ is an orthonormal seq. then any of its subseq. is also an orthonormal seq.

We try to find a "maximal" seq of orthonormal vectors $\{u_j\}_{j=1}^{\infty}$ so that any vector $x \in X$ can be represented as $x = \sum_{j=1}^{\infty} x_j u_j$. Fine if $\dim X < \infty$

What if $\dim X = \infty$?

② $X = L^2(0,\pi)$ $\{\sqrt{\frac{2}{\pi}}\sin nx\}$: orthonormal.

Fact: If $u_1, u_2, \dots, u_n, \dots$ are orthonormal then any subseq. is also orthonormal.

In a finite-dimensional space, say, $\mathbb{R}^n$, we have a set of "complete" orthonormal vectors $e_1, e_2, \dots, e_n$ (each $e_j$ has $j$th component 1 and other components 0), in the sense that any vector $u \in \mathbb{R}^n$ can be uniquely expressed as

$$u = u_1 e_1 + u_2 e_2 + \dots + u_n e_n$$

These are coordinates of $u$.

What about an infinitely dimensional Hilbert space?

$\langle a, b\rangle = \sum_{j=1}^{\infty} a_j b_j$

<u>Example</u> $\ell^2 = \{(a_1, a_2, a_3, \dots) : \text{all } a_j \in \mathbb{R},\ j \geq 1 \text{ and } \sum_{j=1}^{\infty} a_j^2 < \infty\}$

This is a (real) Hilbert space. Let

$e_1 = (1, 0, 0, \dots)$
$e_2 = (0, 1, 0, \dots)$
$e_3 = (0, 0, 1, 0, \dots)$
$\dots\dots$

$\{e_j\}_{j=1}^{\infty}$ is an orthonormal system. This is in fact "complete" as every $u = (a_1, a_2, a_3, \dots) \in \ell^2$

can be expressed as

$$u = \sum_{j=1}^{\infty} a_j e_j \overset{\text{definition}}{=\!=} \lim_{N\to\infty} \sum_{j=1}^{N} a_j e_j$$

check: $\sum_{j=1}^{N} a_j e_j = a_1 e_1 + a_2 e_2 + \cdots + a_N e_N$

$$= (a_1, a_2, a_3, \cdots a_N, 0, 0, 0, \cdots)$$

$$\left\| \sum_{j=1}^{N} a_j e_j - u \right\| = \| (0, 0, \cdots, 0, a_{N+1}, a_{N+2}, \cdots) \|$$

$$= \sum_{j=N+1}^{\infty} a_j^2 \longrightarrow 0 \text{ as } N \to \infty$$

since $\sum_{j=1}^{\infty} a_j^2 < \infty$.

<u>Definition</u> Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal system in a Hilbert space $X$. This system is <u>complete</u> if for any $x \in X$,

$$\langle x, u_j \rangle = 0 \text{ for all } j \geq 1 \implies x = 0.$$

This means that there is no <u>nonzero</u> vectors that are orthogonal to <u>all</u> $u_j$ $(j \geq 1)$.

<u>Theorem</u> Let $X$ be a Hilbert space. Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal system of $X$.

The following are equivalent:

(1) $\{u_j\}_{j=1}^{\infty}$ is complete;

(2) $\overline{\text{span}\{u_j\}_{j=1}^{\infty}}$ is dense in $X$; i.e. $\overline{\text{span}\{u_j\}_{j=1}^{\infty}} = X$.

(3) (Parseval relation) $\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 = \|x\|^2 \quad \forall x \in X$.

(4) $\sum_{j=1}^{\infty} \langle x, u_j \rangle \overline{\langle y, u_j \rangle} = \langle x, y \rangle \quad \forall x, y \in X$

(5) $x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j \quad \forall x \in X.$

<u>Sketch of Proof</u>

(1) $\Rightarrow$ (2) If $\text{span}\{u_j\}_{j=1}^{\infty}$ is not dense in $X$ then $M = \overline{\text{span}\{u_j\}_{j=1}^{\infty}}$ is a proper, closed subspace of $X$. $\exists x_0 \in X \setminus M$. Let $u_0 \in M$ be the best approximation of $x_0$ in $u_0$. Then $x_0 - u_0 \neq 0$, $x_0 - u_0 \perp M$. Hence, $x_0 - u_0 \perp$ all $u_j$ $(j \geq 1)$ $\Rightarrow \{u_j\}_{j=1}^{\infty}$ is not complete. A contradiction.

(2) $\Rightarrow$ (1) Suppose $x \in X$, $\langle x, u_j \rangle = 0$ $\forall j \geq 1$. Since $\text{span}\{u_j\}_{j=1}^{\infty}$ is dense in $X$, $\exists v_k \in \text{span}\{u_j\}_{j=1}^{\infty}$ $(k = 1, 2, \cdots)$ s.t. $v_k \to x$ as $k \to \infty$. Since $\langle x, u_j \rangle = 0$ $\forall j \geq 1$, we have $\langle v_k, x \rangle = 0$. Hence $\langle x, x \rangle = \lim_{k \to \infty} \langle v_k, x \rangle = 0$. $x = 0$. Thus $\{u_j\}_{j=1}^{\infty}$ is complete.

(3) $\Rightarrow$ (1) Obvious (by definition of completeness).

(2) $\Rightarrow$ (3) $\forall x \in X$ $d_N \equiv \text{dist}(x, \text{span}\{u_j\}_{j=1}^{N}) \to 0$ as $N \to \infty$, since $\text{span}\{u_j\}_1^{\infty}$ is dense in $X$.

But, $d_N = \| x - \sum_{j=1}^{N} \langle x, u_j \rangle u_j \|$.

$x - \sum_{j=1}^{N} \langle x, u_j \rangle u_j \perp u_k$ $(1 \leq k \leq N)$. $\|x\|^2 = \|x - \sum_1^N \langle x, u_j \rangle u_j\|^2 + \sum_1^N |\langle x, u_j \rangle|^2$

Hence, $d_N^2 = \|x\|^2 - \sum_1^N |\langle x, u_j \rangle|^2$

So, $d_N \to 0 \iff \sum_1^{\infty} |\langle x, u_j \rangle|^2 = \|x\|^2$.

(4) $\Leftrightarrow$ (3).

(5) $\Rightarrow$ (1) Obvious.

(1) $\Rightarrow$ (5) See (2) $\Rightarrow$ (3) <u>Q.E.D.</u>

This page should be in Section 3.1



<u>Definition</u> Let $X$ be a Hilbert space. A sequence $\{x_k\}_{k=1}^{\infty}$ converges weakly to $x$ in $X$ if all $x_k \in X$ $(k \geq 1)$, $x \in X$, and

$$\langle x_k, u \rangle \longrightarrow \langle x, u \rangle \quad \forall u \in X.$$

<u>Example</u> In $X = L^2(a,b)$. $f_k \to f$ weakly means

$$\lim_{k \to \infty} \int_a^b f_k(x)\, g(x)\, dx = \int_a^b f(x)\, g(x)\, dx \quad \forall g \in L^2(a,b).$$

<u>Thm</u>. Let $X$ be a Hilbert space.

~~(1) If $\{u_j\}_{j=1}^{\infty}$ is a sample~~

(1) $x_k \to x$ weakly in $X \Longrightarrow \{\|x_k\|\}$ is bounded and $\liminf_{k \to \infty} \|x_k\| \geq \|x\|$.

(2) $x_k \to x$ (strongly. i.e., $\|x_k - x\| \to 0$)

$\Longleftrightarrow$ $x_k \to x$ weakly and $\|x_k\| \to \|x\|$.

<u>Proof</u> (1) $\langle x_k, x \rangle \longrightarrow \langle x, x \rangle = \|x\|^2$

$$\|x\|^2 = \lim_{k \to \infty} \langle x_k, x \rangle = \liminf_{k \to \infty} \langle x_k, x \rangle$$
$$\leq \liminf_{k \to \infty} \|x_k\| \, \|x\|.$$

[ Weak convergence of $\{x_k\}$ implies the (strong) boundedness of $\{\|x_k\|\}$. By the Banach-Steinhaus Principle of uniform boundedness (of functionals/operators). ]

(2) $\|x_k - x\|^2 = -\|x_k + x\|^2 + 2\|x_k\|^2 + 2\|x\|^2$

$\|x_k + x\|^2 = \|x_k\|^2 + 2\langle x_k, x \rangle + \|x\|^2$ ($X$: real)

$$\longrightarrow \|x\|^2 + 2\langle x, x \rangle + \|x\|^2 = 4\|x\|^2$$

So, $\|x_k - x\|^2 \to 0$. <u>Q.E.D.</u>

## §3.4 Orthogonal Polynomials

$\langle f, g\rangle = \int_a^b f(x)g(x)dx$

Consider $L^2(a,b)$, a (real) Hilbert space. Denote

$\mathcal{P} = \{$ all (real) polynomials $\}$

$\mathcal{P}_n = \{$ all (real) polynomials of deg. $\le n\}$

Facts: ① $\mathcal{P}_n = \text{span}\{1, x, x^2, \cdots, x^n\}$

② $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n \subset \cdots \subset C([a,b]) \subset L^2(a,b)$

$C([a,b]) = \{$ all (real) continuous functions on $[a,b]\}$.

Best approximation / Least-squares approximation

Given $f \in L^2(a,b)$ and $n \ge 0$ (integer). Find

$$p \in \mathcal{P}_n \text{ s.t. } \|p - f\| = \min_{q \in \mathcal{P}_n} \|q - f\|$$

<u>Solution</u> $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$\langle p, x^j\rangle = \langle f, x^j\rangle \quad (j = 0, 1, \cdots, n)$

$$Ga = f \qquad G = G(1, x, \cdots, x^n) = \left[\langle x^i, x^j\rangle\right]_{i,j=0}^{n}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad f = \begin{bmatrix} \langle f, x^0\rangle \\ \langle f, x\rangle \\ \vdots \\ \langle f, x^n\rangle \end{bmatrix}$$

$G$ is symmetric positive definite. $\Longrightarrow$ unique soln $a$.

i.e., unique best approximation $p$.

Now, let's think about replacing $1, x, \cdots, x^n, \cdots$ by orthonormal polynomials $Q_0(x), Q_1(x), Q_2(x), \cdots$ such that $\deg Q_n = n$ (exactly).

Then $\mathcal{P}_n = \text{span}\{Q_0, Q_1, \cdots, Q_n\}$

$$p(x) = b_0 Q_0(x) + b_1 Q_1(x) + \cdots + b_n Q_n(x)$$

$$\langle p, Q_j \rangle = \langle f, Q_j \rangle \implies \boxed{b_j = \langle f, Q_j \rangle, \quad j = 0, 1, \cdots, n}$$

$$G = I.$$

We can use the Gram-Schmidt orthogonalization process to obtain $Q_0, Q_1, Q_2, \cdots$ starting from $1, x, x^2, \cdots$.

$1 \longrightarrow Q_0 = \frac{1}{\|1\|} 1$ $\qquad$ $\text{span}\{Q_0\} = \mathcal{P}_0 = \text{span}\{1\}$

$1, x \longrightarrow Q_1$. $\quad \langle Q_i, Q_j \rangle = \delta_{ij}$ $\qquad$ $\text{span}\{Q_0, Q_1\} = \mathcal{P}_1 = \text{span}\{1, x\}$

$1, x, x^2 \longrightarrow Q_2$ $\quad \langle Q_i, Q_j \rangle = \delta_{ij}$ $\qquad$ $\text{span}\{Q_0, Q_1, Q_2\} = \mathcal{P}_2 = \text{span}\{1, x, x^2\}$

$\cdots$ $\qquad$ $\cdots$

You don't need to normalize it!

<u>Definition</u> We call a sequence of polynomials $Q_0, Q_1, Q_2, \cdots Q_n, \cdots$ orthogonal polynomials in $L^2(a,b)$ if

(1) $\deg Q_n = n \quad (n = 0, 1, 2, \cdots)$

(2) They are orthogonal w.r.t. the inner product in $L^2(a,b)$.

Note that (1) $\Rightarrow$ $\mathcal{P}_n = \text{span}\{Q_0, Q_1, \cdots, Q_n\}$.
Note also that uniqueness is up to constant multiplication. [This requires a proof.]

<u>Theorem</u> Let $\{Q_n\}_0^\infty$ be orthogonal polynomials in $L^2(a,b)$.

(1) Let $\tilde{Q}_n = c_n Q_n$ so that $\tilde{Q}_n(x)$ has the leading coefficient 1. Then

$$\|\tilde{Q}_n\| = \min_{q \in \mathcal{P}_{n-1}} \|q - x^n\|.$$

(2) $\tilde{Q}_n$ is unique.

(3) Each $Q_n$ $(n \geq 1)$ has exactly $n$ distinct roots in $(a,b)$.

<u>pf</u> (1) $\tilde{Q}_n = x^n - \tilde{q}_{n-1}$ $\quad \tilde{q}_{n-1} \in \mathcal{P}_{n-1}$

$x^n - \tilde{q}_{n-1} \perp \mathcal{P}_{n-1}$ since $\tilde{Q}_n \perp Q_0, Q_1 \cdots Q_{n-1}$

So, $\|\tilde{Q}_n\| = \|x^n - \tilde{q}_{n-1}\| = \min_{q \in \mathcal{P}_{n-1}} \|x^n - q\|$.

(2) By (1).

(3). By induction $\int_a^b Q_n(x) \cdot Q_0 \, dx = 0$

$\Longrightarrow$ $Q_n$ changes sign once (at least).

If $Q_n(x) = (x - x_1) \cdots (x - x_k) p(x)$

$a < x_1 < \cdots < x_k < b$ $\quad \deg p = n - k$.

Then consider $\int_a^b Q_n(x) \, p(x) (x - x_1) \cdots (x - x_k) = 0$.

<u>Q.E.D.</u>

## <u>Legendre Polynomials</u>

# Legendre Polynomials

**Definition** $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right], \quad n = 0, 1, 2, \cdots$$

$\{P_n\}$ are orthogonal polynomials in $L^2(-1,1)$, normalized by
$$P_n(1) = 1 \quad (n = 0, 1, 2, \cdots).$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$
$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$
$$\cdots\cdots$$

**Theorem** (Properties of Legendre Polynomials)

(1) Orthogonality:
$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

(2) Recurrence
$$P_0(x) = 1$$
$$P_1(x) = x$$
$$(n+1) P_{n+1}(x) - (2n+1)\, x P_n(x) + n P_{n-1}(x) = 0$$
$$(n = 1, 2, \cdots)$$

(3) Zeros. For each $n \geq 1$, $P_n(x)$ has $n$ distinct roots in $(-1,1)$.

(4) $P_n(1) = (-1)^n P_n(-1) = 1 \quad (n = 0, 1, 2, \cdots)$

(5) $P_n(x)$ satisfies

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0.$$

## Some applications

**Least-squares approximation.** Let $f \in L^2(-1,1)$.

There exists a unique $p_n \in \mathcal{P}_n$ such that

$$\|p_n - f\| = \min_{q \in \mathcal{P}_n} \|q - f\|.$$

$$p_n = a_0 P_0 + a_1 P_1 + \cdots + a_n P_n$$

Legendre polynomials

Gram matrix $G(P_0, \cdots, P_n) = \begin{bmatrix} \langle P_0, P_0 \rangle & & 0 \\ & \langle P_1, P_1 \rangle & \\ & & \ddots \\ 0 & & \langle P_n, P_n \rangle \end{bmatrix}_{(n+1)\times(n+1)}$

diagonal.

So, $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \sum_{j=0}^{n} \langle P_j, P_j \rangle^{-1} \langle f, P_j \rangle$

$$\boxed{p_n = \sum_{j=0}^{n} \frac{\langle f, P_j \rangle}{\langle P_j, P_j \rangle} P_j}$$

**Numerical integration** Roots of $P_n(x)$.