Section 3.3 Orthogonal Systems, and Orthogonal Polynomials

**Definition** Let $X$ be a Hilbert space. Suppose $u_1, u_2, \ldots, u_n, \ldots$ are vectors of $X$ such that
\[
\langle u_j, u_k \rangle = \delta_{jk}, \quad (j, k = 1, 2, \ldots)
\]
Then we call $\{u_j\}_{j=1}^{\infty}$ an orthonormal sequence of vectors or orthonormal system.

**Example**
\[
X = L^2(-\pi, \pi).
\]

\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{2\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{2\pi}} \sin 2x, \ldots \right\}
\]
is an orthonormal system.

Since
\[
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m \neq 0, \\ 4\pi & \text{if } n = m = 0, \end{cases}
\]
\[
\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m \neq 0, \end{cases}
\]
\[
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0.
\]

If $\{u_j\}_{j=1}^{\infty}$ is an orthonormal seq. then any of its subseq. is also an orthonormal seq.

We try to find a "maximal" seq of orthonormal vectors $\{u_j\}_{j=1}^{\infty}$ so that any vector $x \in X$ can be represented as $x = \sum_{j=1}^{\infty} x_j \cdot u_j$. Fine if $\dim x < \infty$.

What if $\dim x = \infty$?
\[ X = L^2(0, \pi) \setminus \{ \sqrt{\frac{2}{\pi}} \sin n \pi \} : \text{orthonormal} \]

Fact: If \( u_1, u_2, \ldots, u_n, \ldots \) are orthonormal, then any subseq. is also orthonormal.

In a finite-dimensional space, say, \( \mathbb{R}^n \), we have a set of "complete" orthonormal vectors \( e_1, e_2, \ldots, e_n \) (each \( e_j \) has \( j \)th component 1 and other components 0), in the sense that any vector \( u \in \mathbb{R}^n \) can be uniquely expressed as

\[ u = u_1 e_1 + u_2 e_2 + \cdots + u_n e_n \]

These are coordinates of \( u \).

What about an infinitely dimensional Hilbert space?

Example \( L^2 = \{(a_1, a_2, a_3, \ldots) : \text{all } a_j \in \mathbb{R}, j \geq 1 \text{ and } \sum_{j=1}^{\infty} |a_j|^2 < \infty \} \)

This is a (real) Hilbert space. Let

\[ e_1 = (1, 0, 0, \ldots) \]
\[ e_2 = (0, 1, 0, \ldots) \]
\[ e_3 = (0, 0, 1, 0, \ldots) \]

\( \{e_j\}_{j=1}^{\infty} \) is an orthonormal system. This is in fact "complete" as every \( u = (a_1, a_2, a_3, \ldots) \in L^2 \)
can be expressed as
\[
u = \sum_{j=1}^{\infty} a_j e_j = \lim_{N \to \infty} \sum_{j=1}^{N} a_j e_j
\]
check: \[\frac{1}{N} \sum_{j=1}^{N} a_j e_j = a_1 e_1 + a_2 e_2 + \cdots + a_N e_N \]
\[
\| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j e_j - u \| = \| (0, a_1, a_2, \ldots, a_N, 0, 0, \ldots) \|
\]
\[
= \sqrt{\sum_{j=N+1}^{\infty} a_j^2} \to 0 \text{ as } N \to \infty
\]
since \[\sum_{j=1}^{\infty} a_j^2 < \infty .
\]

Definition Let \{u_j\}_{j=1}^{\infty} be an orthonormal system in a Hilbert space \(X\). This system is complete if for any \(x \in X\),
\[\langle x, u_j \rangle = 0 \text{ for all } j \geq 1 \implies x = 0 .
\]
This means that there is no nonzero vector that are orthogonal to all \(u_j\) \((j \geq 1)\).

Theorem Let \(X\) be a Hilbert space. Let \(\{u_j\}_{j=1}^{\infty}\) be an orthonormal system of \(X\).
The following are equivalent:
1. \(\{u_j\}_{j=1}^{\infty}\) is complete;
2. \(\text{span}\{u_j\}_{j=1}^{\infty}\) is dense in \(X\);
3. (Parseval relation) \[\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 = \|x\|^2 \quad \forall x \in X .\]
(4) \[ \sum_{j=1}^{\infty} \langle x, y_j \rangle \langle y, y_j \rangle = \langle x, y \rangle \quad \forall x, y \in X \]

(5) \[ x = \sum_{j=1}^{\infty} \langle x, y_j \rangle y_j \quad \forall x \in X. \]

**Sketch of Proof**

(1) \(\Rightarrow\) (2) If \(\text{span} \{y_j\}_{j=1}^{\infty}\) is not dense in \(X\), then \(M = \text{span} \{y_j\}_{j=1}^{\infty}\) is a proper closed subspace of \(X\). \(\exists x_0 \in X \setminus M\). Let \(u_0 \in M\) be the best approximation of \(x_0\) in \(M\). Then \(x_0 - u_0 \neq 0,\) \(x_0 - u_0 \perp M\)

Hence, \(x_0 - u_0 \perp \text{all } y_j. (j \geq 1) \Rightarrow \{y_j\}_{j=1}^{\infty}\) is not complete. A contradiction.

(2) \(\Rightarrow\) (1) Suppose \(x \in X, \langle x, y_j \rangle = 0 \quad \forall j \geq 1\).

Since \(\text{span} \{y_j\}_{j=1}^{\infty}\) is dense in \(X\), \(\exists x_k \in \text{span} \{y_j\}_{j=1}^{\infty}\)

\((k = 1, 2, \ldots) \) s.t. \(x_k \to x, \) as \(k \to \infty.\)

Since \(\langle x_k, y_j \rangle = 0\)

\(\forall j \geq 1\), we have \(\langle x_k, x \rangle = 0\).

Hence \(\langle x, x \rangle = \lim_{k \to \infty} \langle x_k, x \rangle = 0\).

\(x = 0.\) Thus \(\{y_j\}_{j=1}^{\infty}\) is complete.

(3) \(\Rightarrow\) (1) Obvious (by definition of completeness).

(2) \(\Rightarrow\) (3) \(\forall x \in X, d_x \equiv \text{dist} (x, \text{span} \{y_j\}_{j=1}^{\infty}) \to 0\)

as \(N \to \infty.\) Since \(\text{span} \{y_j\}_{j=1}^{\infty}\) is dense in \(X.\)

But, \(d_x = \| x - \frac{1}{N} \sum_{j=1}^{N} \langle x, y_j \rangle y_j \|.

\(x - \frac{1}{N} \sum_{j=1}^{N} \langle x, y_j \rangle y_j \perp u_k \quad (k \in N).\)

\(\| x \| = \| x - \frac{1}{N} \sum_{j=1}^{N} \langle x, y_j \rangle y_j \| + \frac{1}{N} \sum_{k=1}^{N} \| y_k \|^2\)

Hence, \(d_x^2 = \| x \|^2 - \frac{1}{N} \sum_{j=1}^{N} | \langle x, y_j \rangle |^2 + \frac{1}{N} \sum_{k=1}^{N} \| y_k \|^2\)

So, \(d_x \to 0 \iff \sum_{j=1}^{\infty} | \langle x, y_j \rangle |^2 = \| x \|^2.

(4) \(\subseteq\) (3).

(5) \(\Rightarrow\) (1) Obvious.

(1) \(\Rightarrow\) (5) See (2) \(\Rightarrow\) (3) \(\quad \square.\)
**Definition** Let \( X \) be a Hilbert space. A sequence \( \{x_k\}_{k=1}^{\infty} \) converges weakly to \( x \) in \( X \) if all \( x \in X \), and
\[
\langle x_k, u \rangle \to \langle x, u \rangle \quad \forall u \in X
\]

**Example** In \( X=L^2(a,b) \), \( f_k \to f \) weakly means
\[
\lim_{k \to \infty} \int_a^b f_k(x) g(x) \, dx = \int_a^b f(x) g(x) \, dx \quad \forall g \in L^2(a,b).
\]

**Thm.** Let \( X \) be a Hilbert space.

1. If \( \{x_j\}_{j=1}^{\infty} \) is a sequence
   - \( x_j \to x \) weakly in \( X \) \( \Rightarrow \) \( \{\|x_j\|\} \) is bounded and \( \lim \inf_k \|x_k\| \geq \|x\| \).

2. \( x_j \to x \) (strongly, i.e., \( \|x_j-x\| \to 0 \)) \( \iff \) \( x_j \to x \) weakly and \( \|x_j\| \to \|x\| \).

**Proof**

1. \( \langle x_k, x \rangle \to \langle x, x \rangle = \|x\|^2 \)
\[
\|x\|^2 = \lim_{k \to \infty} \langle x_k, x \rangle \geq \lim \inf_{k \to \infty} \langle x_k, x \rangle
\]

Weak convergence of \( \{x_k\} \) implies the (strong) boundedness of \( \{x_k\} \). By the Banach-Steinhaus principle of uniform boundedness (of functionals).

2. \( \|x_k-x\|^2 = -\|x_k+x\|^2 + 2\|x\|^2 + 2\|x\|^2 \quad \|x_k+x\|^2 = \|x_k\|^2 + 2\langle x_k, x \rangle + \|x\|^2 \)
\[
\to \|x\|^2 + 2\langle x, x \rangle + \|x\|^2 = 4\|x\|^2
\]
\[
\Rightarrow \|x_k-x\|^2 \to 0.
\]

\( \Box \)
§3.4 Orthogonal Polynomials

Consider $L^2(a,b)$, a (real) Hilbert space. Define

$P = \{ \text{all (real) polynomials} \}$

$P_n = \{ \text{all (real) polynomials of deg. \leq n} \}$

Facts:
1. $P_n = \text{span} \{ 1, x, x^2, \ldots, x^n \}$
2. $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \subseteq \ldots \subseteq C([a,b]) = L^2(a,b)$
   $C([a,b]) = \{ \text{all (real) continuous functions on } [a,b] \}$.

Best approximation / Least-squares approximation

Given $f \in L^2(a,b)$ and $n \geq 0$ (integer). Find

$p \in P_n \text{ s.t. } \| p - f \| = \min_{q \in P_n} \| q - f \|$

Solution

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$<p, x^j> = <f, x^j> \quad (j = 0, 1, \ldots, n)$

$G a = f$ \quad $G = G(l, x, \ldots, x^n) = \begin{bmatrix} <x^0, x^0> & \cdots & <x^0, x^n> \\ \vdots & \ddots & \vdots \\ <x^n, x^0> & \cdots & <x^n, x^n> \end{bmatrix}$

$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$ \quad $f = \begin{bmatrix} <f, x^0> \\ <f, x^1> \\ \cdots \\ <f, x^n> \end{bmatrix}$

$G$ is symmetric positive definite. \implies$unique solution $a$, i.e., unique best approximation $p$.

Now, let's think about replacing $1, x, \ldots, x^n$ by orthonormal polynomials $Q_0(x), Q_1(x), Q_2(x), \ldots$ such that $\deg Q_n = n$ (exactly).
Then \( P_n = \text{span} \{ q_0, q_1, \ldots, q_n \} \),
\[ P(x) = b_0 q_0(x) + b_1 q_1(x) + \cdots + b_n q_n(x) \]
\[ \langle p, q_j \rangle = \langle f, q_j \rangle \implies \left[ b_j = \langle f, q_j \rangle, \ j = 0, \ldots, n \right] \]
\[ G = I. \]

We can use the Gram-Schmidt orthogonalization process to obtain \( q_0, q_1, q_2, \ldots \) starting from 1, \( x, x^2, \ldots \).

1 \to Q_0 = \frac{1}{\|1\|} 1
1, x \to Q_1 \quad \langle Q_0, Q_1 \rangle = \langle 1, x \rangle
1, x, x^2 \to Q_2 \quad \langle Q_0, Q_2 \rangle = \langle 1, x^2 \rangle
1, x, x^2, \ldots

You don't need to normalize it!

**Definition** We call a sequence of polynomials \( q_0, q_1, q_2, \ldots, q_n, \ldots \) orthogonal polynomials in \( L^2(a, b) \) if

1. \( \deg q_n = n \) \( (n = 0, 1, 2, \ldots) \)
2. They are orthogonal w.r.t. the inner product in \( L^2(a, b) \).

Note that (1) \( \implies P_n = \text{span} \{ q_0, q_1, \ldots, q_n \} \).

Note also that uniqueness is up to constant multiplication. [This requires a proof.]
Theorem 3. Let \( \{ \tilde{q}_n \} \) be orthogonal polynomials in \( L^2(a, b) \).

1. Let \( \tilde{q}_n = c_n q_n \) so that \( \tilde{q}_n(x) \) has the leading coefficient 1. Then
   \[ ||\tilde{q}_n|| = \min_{\tilde{q}_n \in \mathbb{P}_n} ||q - x^n||. \]

2. \( \tilde{q}_n \) is unique.

3. Each \( \tilde{q}_n \) (\( n \geq 1 \)) has exactly \( n \) distinct roots in \( (a, b) \).

\[ \text{PF} \]

1. \( \tilde{q}_n = x^n - \tilde{p}_n \), \( \tilde{p}_n \in \mathbb{P}_{n-1} \)
   \( x^n - \tilde{p}_n \perp \mathbb{P}_{n-1} \) since \( \tilde{q}_n \perp q_0, q_1, \ldots, q_{n-1} \)
   So, \( ||\tilde{p}_n|| = \min_{\tilde{q}_n \in \mathbb{P}_n} ||x^n - \tilde{q}_n|| = ||x^n - \tilde{p}_n|| \)

2. By (1).

3. By induction \( \int_a^b \tilde{q}_n(x) Q_0 \, dx = 0 \)
   \[ \Rightarrow \tilde{q}_n \text{ changes sign exactly (at least)}. \]
   If \( \tilde{q}_n(x) = (x - x_1) \cdots (x - x_k) y(x) \)
   \( a < x_1 < \cdots < x_k < b \) \( \deg y = n-k \)
   Then consider \( \int_a^b \tilde{q}_n(x) Q_{n-k} (x - x_1) \cdots (x - x_k) = 0 \). \( \Box \).
Legendre Polynomials

**Definition** \( P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n], \quad n = 0, 1, 2, \ldots \)

\{P_n\} are orthogonal polynomials in \( L^2(-1,1)\), normalised by \( P_n(1) = 1 \) \( (n = 0, 1, 2, \ldots) \).

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{3}{2} x^2 - \frac{1}{2} \]
\[ P_3(x) = \frac{5}{2} x^3 - \frac{3}{2} x \]
\[ P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8} \]
\[ \ldots \]

**Theorem (Properties of Legendre Polynomials)**

1. **Orthogonality:**
   \[ \int_{-1}^{1} P_m(x) P_n(x) \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases} \]

2. **Recurrence**
   \[ P_0(x) = 1 \]
   \[ P_1(x) = x \]
   \[ (n+1) P_{n+1}(x) - (2n+1) x P_n(x) + n P_{n-1}(x) = 0 \]
   \( (n = 1, 2, \ldots) \)
3) Zeros. For each \( n \geq 1 \), \( P_n(x) \) has \( n \) distinct roots in \((-1, 1)\).

4) \( P_n(1) = (-1)^n P_n(-1) = 1 \) \((n = 0, 1, 2, \ldots)\).

5) \( P_n(x) \) satisfies
\[
(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0.
\]

Some applications

**Least-squares approximation.** Let \( f \in L^2((-1, 1)) \). There exists a unique \( P_n \in P_n \) such that
\[
\|P_n - f\| = \min_{g \in P_n} \|g - f\|.
\]

\[
P_n = a_0 P_0 + a_1 P_1 + \cdots + a_n P_n
\]

Legendre polynomials

Gram matrix \( G(P_0, \ldots, P_n) = \begin{bmatrix} \langle P_0, P_0 \rangle & \langle P_0, P_1 \rangle & \cdots & \langle P_0, P_n \rangle \\ \langle P_1, P_0 \rangle & \langle P_1, P_1 \rangle & \cdots & \langle P_1, P_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle P_n, P_0 \rangle & \langle P_n, P_1 \rangle & \cdots & \langle P_n, P_n \rangle \end{bmatrix} \)

So,
\[
\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} \langle f, P_j \rangle \end{bmatrix} \begin{bmatrix} \langle P_j, P_j \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle P_j, P_j \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle f, P_j \rangle \end{bmatrix}
\]

\[
P_n = \sum_{j=0}^{n} \frac{\langle f, P_j \rangle}{\langle P_j, P_j \rangle} P_j
\]

**Numerical integration.** Roots of \( P_n(x) \).